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Market Efficiency and Option Pricing with Gaussian Term Structure of Interest Rates

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CONTENTS

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1 Introduction

The main body of my dissertation consists of three chapters (chapter 2 through chapter 4) each dealing with a different topic yet all related to the notion of efficiency of the financial markets which plays a central role in asset pricing.

Chapter 2 investigates into the relationship between changes in overall market friction and changes in asset prices. Under the rather broad interpretation that market friction is the welfare loss to agents due to market deficiencies each time they trade, this chapter examines the structural effects of market friction on security prices and trade volumes in an equilibrium setting where investors are risk-neutral. I present a dynamic equilibrium model solved out in a closed form in which an ex-cost risk-neutral valuation of assets are obtained: the market price of a stock is a linear function of its fundamentals. All friction effects are contained in the coefficient. I find that if the market friction is structurally biased in buyer's favor in the sense that buyers incur marginal friction that increases (or decreases) at a slower pace than sellers do when both sides try to readjust their sizes of trade in the face of a shock in market friction, an increase (decrease) in the friction parameter in a given trading period will drive both the price and its volatility down (up). On the other hand, if the market friction is biased in sellers' favor, then an increase (decrease) in the friction parameter in a given period will push the prices up (down) along with its volatility. If the friction structure is "balanced", then there is no friction parameter effects on prices (and their volatility). Hence in the case of structural bias in favor of the buyers, a steady reduction in the friction parameters may result in a steady increase of equity prices as well as price volatility. In any case, trade volume falls in response to an increase in market friction.

Chapter 3 generalizes the continuous-time asset market beyond the traditional framework of *Brownian* motion driven stock prices by replacing the *Brownian* motion process as the fundamental risk generating factors with square-integrable continuous martingales. It is found that markets that consist of a bond and equities are still efficient in the sense that markets are dynamically complete, risk-neutral valuation holds under some martingale measure and the markets are free of arbitrage.

Chapter 4 generalizes the *Black-Schole's* option pricing model by considering interest rate risks. I incorporate general *Gaussian* term structure into the short rate process and develops closed-form formulas of equity option valuation as well as bond prices with different maturities which define the term structure of economy-wide interest rates. By allowing free form of the coefficient functions in the linear stochastic differential equation that defines the short rate process, the popular models such as the *Vasicek* model, the *Ho-Lee* model as well as the *Hull-White* model are covered as special cases in my treatment.

2 Structural Effects of Market Friction on Asset Prices

2.1 Introduction

Because of the empirical price anomalies associated with many representative agent models in asset pricing theory which is obtained in a perfect security trading environment, many people tried to explain the seemingly poor empirical performances of CCAPM by the various kinds of market deficiencies such as transactions costs, market incompleteness and information asymmetry. Research in this direction has turned up a sizable literature both theoretical and empirical. See for example, *J. Heaton* and *D. Lucas* (1994), *E. Luttmer* (1994), *Jiang Wang* (1993), *M. Jackson* and *J. Peck* (1994), *J. Bradford De Long et al* (1990) and *Jie Hu* (1996).

Since the various forms of market deficiencies seem to be responsible for observed price anomalies, the common wisdom among financial economists is that as markets become more efficient in terms of improved overall market efficiency, one should observe a behavior of asset prices more in line with classical fundamental valuation in an entirely frictionless market framework such as CCAPM.

However, casual empiricism seems to suggest otherwise. The past two decades have witnessed a phenomenal change in the financial markets. Transaction fees have decreased greatly due to increased competition among brokerage services offered by both the boutique shops and the full service bulge bracket firms. For example, *John Marshall* and *E.M. Ellis* (1994) report that the cost of transacting for institutional traders on a wholesale scale have dropped to 1/20 of the levels prevailing twenty years ago. Also, thanks to advances in microchip technology, costs of information search have greatly reduced and instant executions of trade become increasingly commonplace. Moreover, markets are made more "complete" as more products become available ranging from plain vanilla to the most exotic, thanks to the ingenuity of financial engineering. (see *Merton Miller* 1994). And yet, there is no empirical evidence that indicates reduced price anomalies. On the other hand, the bulk of current research also fail to adequately address the issue of what impact it will have on asset prices as markets more and more "friction-free". Motivated by this observation, my paper tries to answer, in a general equilibrium framework, the question under what conditions an improvement in the overall market efficiency may actually cause an increase in the degree of these price anomalies, and under what conditions the relation is exactly the opposite.

My paper differs in methodology from the existing research on market deficiencies in a number of aspects. First, instead of looking at specific types of deficiencies, I pool together all the existing institutional, physical and psychological barriers investors (sellers and buyers alike) will have to overcome each time they engage in trade. Indeed the conceptual distinctions among the various kinds of deficiencies are often blurred at least empirically. Transactions costs may be interpreted as one form of market incompleteness or, as is noted in *D. Lucas* (1992), may exacerbate the incompleteness effects, and information asymmetry can induce information search costs, etc.. Thus, although the terms market friction and transactions costs are used, my paper does not address any specific kind of financial costs involved in transacting activities. Rather it is on market deficiencies in general and the terms are therefore given a much broader and richer interpretation. The literal transaction fees (which may justify proportional cost functions, or fixed costs) are only one of the many factors that serve to deter a smooth trade especially when the transaction size is large. The liquidity concerns, the potential cost of search for trade partners or an efficient distribution network, the cost of information to the less informed "outsiders" (as opposed to corporate insiders), the concerns of inability to perfectly reallocate income streams across time and states of nature due to market incompleteness such as borrowing and short sell constraints, may play a far more significant role in an investor's decision. Therefore, I summarize in a function all the welfare loss to agents in trade due to the factors that jointly work against smooth trading activities and call this function the premium of market friction or simply the transactions costs. A decrease in the friction parameters will be interpreted as an improvement in the overall market efficiency and vice versa.

Secondly, my paper distinguishes itself from the bulk of the literature on the effects of transactions costs on asset prices in that the type of cost effects examined is quite different from current research. As is observed in *S. Grossman* and *G. Laroque* (1990), most empirical and theoretical papers on effects of transactions costs are centered around explaining how the presence of costs distorts agents' IMRS (inter-temporal marginal rate of substitution) causing the deviation of asset prices from what is justifiable through CCAPM. I call this type of effects the "effects through pure IMRS distortion" Such an approach seems to suffer some drawbacks. As is observed in *G. Constantinides* (1980 and 1994), effects of this type are of second order especially when the costs are small, which is certainly the case when they are literally interpreted as brokerage fees and commissions, etc.. Moreover, this approach does not appear to adequately account for the paradox that my paper sets out to dissect, namely, why the exacerbation of price anomalies seems to co-exist with the rapid evolution of financial markets toward being more efficient in general. My paper identifies another and perhaps more potent source of cost effects which stems from the inherent structural imbalance of market friction. I call this type of effects the "structural effects" of market friction.

Roughly speaking, structural effects are the type of effects on asset prices that arises when market friction is structured in such a way that aggregate buyers respond differently than aggregate suppliers in the face of shocks in the friction parameters. The intuition behind this is fairly straightforward. Market friction acts like a double-edged sword that cuts into the profits of both the buyers and sellers. But the broad interpretation of market friction suggests that buyers and sellers in a given market may face trade barriers that are quite different in nature. Therefore, in the face of a change in the marginal friction, the two groups of investors will respond differently in their assessments of marginal change of friction premiums, or marginal costs, which will in turn force them to readjust their demand (or supply) of the security accordingly, temporarily causing mismatch of the demand and the supply in the security. Price will have to adjust to put the market back in equilibrium. For instance, when marginal friction reduces, both buyers and sellers have incentives to increase their sizes of trade since both sides find themselves operating at the levels with marginal revenues strictly greater than marginal costs[1]. But buyers' marginal friction may very well increase at a slower rate than sellers and thus aggregate demand for the security expands more than aggregate supply creating a momentary shortage of supply. Price must adjust upwardly to quiet down demand on the one hand and to attract more supply on the other, until the market is back in equilibrium. I refer to this situation as the "structural bias in favor of the buyers".

Although structural bias in sellers' favor may also happen in theory, the former scenario appears to fit in more realistically with the on-going changes in our financial markets. An overall improvement in market efficiency seems to attract more activities from the buyer side than from the sellers. This phenomenon is not without

[1] Here the accounting definition of revenue is the cash flow (including capital gains and dividend earnings) in a given period excluding friction premium. Cost is defined to be the friction premium.

a good explanation. Selling activities primarily come from those who are already in the financial markets, as in the case of liquidity sales. On the other hand, buying activities not only come from those who are already in the markets but also from those who are originally outside the markets. Indeed, advances in information technology nowadays provide an easy access to the necessary information for an average person at a much cheaper rate. A much reduced brokerage fee makes it cheaper for individuals to invest in stocks (and bonds). The availability of more advanced hedging tools such as derivatives products makes an outsider more willing to dip into the financial markets (more often as buyers than as sellers since people don't typically come in and short sell). The rapidly emerging mutual funds make it much easier for individual investors with limited financial resources to diversify their portfolios and spare additional information and monitoring costs by investing in various kinds of growth and income funds and become indirect shareholders. In short, the improvement in overall market efficiency seems to attract market entries by many investors who would otherwise shy away from financial investments, causing more money to be pumped into (rather than taken out of) the markets. With a relatively stable total supply of stocks, this will inevitably drive up market prices. The data contained in table 1 is perhaps revealing of the money driven stock price increase.

Table 1

Assets of the Financial Institutions ($Billion)

Years	Financial Industry	Insurance Companies	Commercial Banks	Mutual Funds
1989	12,152 (100	1,734 (14.2%)	3,233 (26.6%)	555 (4.6%)
1990	12,910 (100%)	1,880 (14.6%)	3,342 (25.9%)	578 (4.5%)
1991	14,784 (100%)	2,092 (14.2%)	3,440 (23.3%)	814 (5.5%)
1992	15,876 (100%)	2,247 (14.1%)	3,640 (22.9%)	1050 (6.6%)

Note: Numbers in brackets indicate relative sizes of the categories (compared within each row). Data source: *Annual Statistical Digest*, Board of Governors.

To isolate the structural effects, I look into a risk-neutral world (see footnote 2), where investors trade securities out of time preference motives. Since in general the observed cost effects are the result of the complex interactions of the two sources, and the ultimate result depends on which is the dominant one, we believe that structural effects of market friction should receive equal treatment, if not more. Indeed, investors (arbitrageurs and speculators alike) are known to have long exploited the cost structural imbalance to their advantages by switching between buyer side and seller side and the structural effects might well become dominant, especially if the effects through IMRS distortions alone are inadequate to explain the My paper finds that it is the inherent structure of market friction rather than the sheer magnitude of costs that contributes to the price anomalies.

A word on the functional forms of market friction is due. Casual empiricism indicates that market friction with this broad interpretation should exhibit convexity with respect to transaction size. In other words, market friction typically entails diseconomy of scale since the average costs increase with transaction sizes[2] . An asset becomes increasingly more difficult to clear out of one's portfolio for liquidity reasons when the volume is large. Large size transactions often require time consuming SEC procedures that are otherwise not necessary. Debt financed purchase of stocks or other types of assets becomes increasingly more difficult when the amount of cash required is large since collateral requirement involves high opportunity costs. Financial institutions are known to charge higher interest rates for large size borrowing in order to offset the risk of default (this is also true of public debt instruments as in the case of high-yield junk bond issues that are often used to finance large size LBO's). Finally, market incompleteness is nicely captured by convexity. People tend to use a set of constraints (as in short sale constraints or borrowing constraints) to describe the scope of allowable portfolio choices. It is my belief that, given the vast investment opportunities, it is not entirely unreasonable to replace the hard, rigid and insurmountable constraints with convexity of cost premium. Punishment at an accelerating rate may tie up an investor's hands just like an artificial constraint (we may refer to this as the shadow price of overcoming the constraints. After all, a constraint is just like a cost function in the utility in the corresponding Lagrangean). One may overdraw one's bank account. But in so doing, he (she)'d better prepare for costs (both in money and the credit track record) at an increasing rate. In other words, convexity here serve to smooth up what could be a

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[2] A counter example might be brokerage fees as discount rates are often available for large size transaction. But again, the influence of literal transaction fees may be small compared to other types of trade barriers.

binding solution (with hard constraints imposed) to an interior one. In short it appears to be a reasonable assumption that investors demand a premium for market friction at an increasing rate as the size of transaction grows since it becomes increasingly more difficult to overcome all the physical, mental and financial barriers to trade[3]. Thus I do not assume specific functional forms for the premium of market friction but I do impose convexity.

The rest of the paper is organized as follows. Section 2 formally presents the equilibrium model together with its dynamic solution concept. Section 3 looks at a special case in which there are two agents trading securities with each other and the cost function is symmetric (as is typically assumed). It alerts us to the fact that market friction actually has no effects on equilibrium prices (and its volatility) other than driving trade volumes down when cost structure treats buyers and sellers in a balanced fashion in that sellers and buyers equally adjust friction premiums whenever there is a shock in the friction parameters. Section 4 studies the structural effects of market friction in a general setting with asymmetric cost functions and multiple traders. Section 5 further explains why friction effects can be decomposed into two parts that are very different in nature, namely structural effects and risk aversion effects. Section 6 is the conclusion.

2.2 Model Setup

The equilibrium concept is the usual competitive one. No game-theoretic micro structure or price formation mechanism is assumed.

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[3] Although technically my model is a risk-neutral one, the convexity of friction premium actually allows for a portion that is the certainty equivalent of one's risky investment, i.e. the risk premium which seems to decrease as markets become more complete in terms of availability of hedging tools such as derivatives products and channels of diversification such as mutual funds. Hence investors are not truly risk neutral.

Underlying uncertainty: $(\Omega, \mathcal{F}, P)$, any probability space.

Time horizon: $\mathsf{T} = \{0, 1, 2, ..., T\}$, where T is a finite integer.

Information: There is no information asymmetry and all investors are equipped with the same information structure described by a filtration $\mathsf{F} = \{\mathcal{F}_t\}_{t\in\mathsf{T}}$ on $(\Omega, \mathcal{F})$. We assume that $\mathcal{F}_0 = \{\Omega, \varnothing\}$ so that a random variable x is $\mathcal{F}_0$-measurable iff x = constant P-a.s.. Also $\cup_{t\in\mathsf{T}}\mathcal{F}_t = \mathcal{F}$.

Securities: There are N securities indexed by $n = 1, 2, ..., N$. Security n has an F-adapted price process $S^{(n)} = \{S_t^{(n)}\}_{t\in\mathsf{T}}$. It also has an F-adapted dividend process $D^{(n)} = \{D_t^{(n)}\}_{t\in\mathsf{T}}$. Security trading in the nth security market involves transaction costs or friction premium, which is represented by a convex function q_n: $\mathbb{R}^N \to \mathbb{R}_+$ such that $q_n(0) = 0$. The argument x in $q_n(x)$ will represent the dollar size of order in security n with positive value so that a positive value of x represents a buy order and a negative value of x represents a sell order.

Strategy A trading strategy is an $\mathbb{R}^N$-valued adapted process $X = \{X_t\}_{t\in\mathsf{T}}$ where X_t is the net portfolio holding in period t. Let Θ be the linear space of trading strategies.

Cash Flow the market is open for trading from period 0 through period $T - 1$. In the final period T, all final date dividends are delivered to stock holders costlessly making all the stocks void for any further trading. Thus an F-adapted cash flow process $c = \{c_t\}_{t\in\mathsf{T}}$ is said to be financed by strategy

$X = \{X_t\}_{t\in T}$ iff the following is satisfied, with X_{-1} being the endowment of portfolio at the very beginning of period 0:

$$c_t = X_{t-1}\cdot D_t - S_t\cdot(X_t - X_{t-1}) - \sum_{n=1}^{N} q_n[S^{(n)}\cdot(X_t^{(n)} - X_{t-1}^{(n)})],\ 0 \le t \le T-1$$

$$c_T = X_{T-1}\cdot D_T$$

Agents: There are m investors indexed by $i = 1, 2, \ldots, m$. Let A denote the set of agents. Agent i has a utility function $U^i(c) = \sum_{t=0}^{T} \delta_i^t\, c_t$ over the set of all adapted cash inflow processes $c = \{c_t\}_{t\in T}$, where δ_i is agent i's time preference parameter.

Subject to the initial portfolio endowment X_{-1}, an agent problem is described by the following mathematical program.

Max 1
$$\max_{(c,X)\in\Lambda(X_{-1})} E\,[\textstyle\sum_{t=0}^{T} \delta_i^t\, c_t]$$

Equilibrium Agents are price takers. An equilibrium is a vector $((c^i, X^i)_{i\in M}, S)$ s.t. for each i, (c^i, X^i) solves **Max 1** with the given price system S and the security market clears:

$$\sum_{i\in A} X^i = 0$$

We will only consider the case in which the agent problem is amenable to the dynamic programming approach. There are N securities traded on the market.

Security 1 through N has a dividend process D whose state space is represented by a nonempty measurable set $Z \subseteq \mathbb{R}^N_{++}$. We assume that D is *Markovian* with transition probability function $P_{t,t+1}$: $Z \times \mathcal{B}(Z) \rightarrow [0, 1]$ s.t. for every $x \in \Omega$ fixed, $P_{t,t+1}(x, \cdot)$ is a probability distribution on $(Z, \mathcal{B}(Z))$, and for every event $A \in \mathcal{B}(Z)$ fixed, $P_{t,t+1}(\cdot, A)$ is measurable. Let $\Omega = Z^T$ be the underlying space of states of nature with $\mathcal{F} = \boxtimes_{t \in \mathrm{T}} \mathcal{F}_t$, where $\mathcal{F}_t = \mathcal{B}(Z)$, $\forall\ t \in \mathrm{T}$. The filtration F is the past history of dividend earnings.

A natural setup of the dynamic problem of a generic agent is as follows. Let $\mathcal{X} = \mathbb{R}^N$ be the endogenous state space so that each $X \in \mathcal{X}$ is regarded as a net portfolio holding. Let Γ_t: $\mathcal{X} \times Z \twoheadrightarrow \Theta$ be a set-valued correspondence given by $\Gamma_t(X, D) = \mathbb{R}^N_+ \times \mathbb{R}^N_+$. A generic agent faces with recursive problem at the beginning of each period t given net portfolio holding X_t and dividend earning D_t: he chooses a portfolio strategy process $X = \{X_t\}_{t=0}^{T-1}$ to maximize the following quantity

$$V_t(X_{t-1}, D_t) = \max_{(X_t) \in \Gamma(X_{t-1}, D_t)} \{X_{t-1} \cdot D_t - S_t \cdot (X_t - X_{t-1}) - \textstyle\sum_{n=1}^N q_n [S^{(n)} \cdot (X_t^{(n)} - X_{t-1}^{(n)})] +$$

$$+ \int_Z V_{t+1}(X_t, D_{t+1}) P_t(D_t, \mathrm{d}D_{t+1}) \}, 0 \le t \le T-1$$

$$V_T(X_{T-1}, D_T) = X_{T-1} \cdot D_T, \qquad \text{subject to } X_{-1} \text{ given as endowment}$$

Investors have incentive to trade because of different time preference rates. We will see that the person with the lowest rate is always a net seller while the person with the highest rate is always a net buyer. Note that because of the presence of convex transaction costs, it is not optimal for sellers to sell all his stocks nor is it optimal for buyers to purchase arbitrarily large number of shares in a single market transaction.

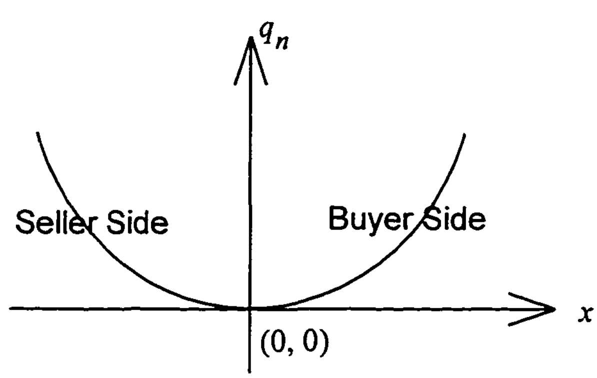
2.3 Symmetric Costs and Two Agents

In this section we study the case in which there are two investors i, j both risk-neutral but with different discount rates δ_i and δ_j (this is crucial to ensure trade in equilibrium). The cost function q_n satisfying the following conditions

A2.3.1 q_n: $\mathbb{R} \rightarrow \mathbb{R}_+$ is strictly convex and differentiable, $\forall$ n;

A2.3.2 $q_n(-x) = q_n(x)$, $\forall$ $x \in \mathbb{R}$, $\forall$ n.

Rem The restriction $q_n(x) = q_n(-x)$ in **A2.3.2** simply says that transaction costs are symmetric w.r.t. purchase and sales. In fact if $q_n(x) = f_n(x^2)$ (where f_n is differentiable), then q_n satisfies the requirement. Being a function of x^2 is equivalent to being a function of $|x|$. The only difference lies in that a differentiable function of $|x|$ may not be differentiable at $x = 0$



the graph of a symmetric friction premium function

Let's consider the static case in which there are two periods $t = 0$ and 1. Markets are open on day 0. There are N stocks traded. Let D_0 be the dividend (a constant vector) delivered on day 0 to investors with initial endowments of portfolio. Let D_1 (a vector-valued random variable) be the dividend to be delivered on day 1. Suppose there are two investors both risk-neutral but with different time preference rates. Let investor i be endowed with portfolio X_i on day 0. Let S be the going market price. Then investor i solves the following problem:

$$\max_{Y_i} \{X_i \cdot D_0 - S \cdot Y_i - \Sigma_n \, \varepsilon_n \cdot q_n[S^{(n)} \cdot (Y_i^{(n)} - X_i^{(n)})] + \delta_i \cdot \int_Z Y_i \cdot D_1 P(\mathrm{d}D_1)\}$$

where $\varepsilon_n > 0$ serves as a friction parameter representing the marginal change in friction premium. Let

$$\mu = \int_Z D_1 P(\mathrm{d}D_1).$$

(Note that μ is an $\mathbb{R}^N$-vector). Now the agent's problem becomes

$$\max_{Y_i} \{X_i \cdot D_0 - S \cdot Y_i - \Sigma_n \varepsilon_n \cdot q_n[S^{(n)} \cdot (Y_i^{(n)} - X_i^{(n)})] + \delta_i \cdot Y_i \cdot \mu\}$$

FOC

$Y_i^{(n)}$: $\quad -S^{(n)} - \varepsilon_n \cdot S^{(n)} \cdot q_n'[S^{(n)} \cdot (Y_i^{(n)} - X_i^{(n)})] + \delta_i \cdot \mu^{(n)} = 0$, or (assuming $S^{(n)} \neq 0$)

$$\varepsilon_n q_n'[S^{(n)}(Y_i^{(n)} - X_i^{(n)})] = \frac{\delta_i \mu^{(n)}}{S^{(n)}} - 1 \Rightarrow$$

$$S^{(n)} \cdot (Y_i^{(n)} - X_i^{(n)}) = r_n[\frac{1}{\varepsilon_n}\left(\frac{\delta_i \mu^{(n)}}{S^{(n)}} - 1\right)], \qquad (2.3.1)$$

where r_n is the inverse of q_n'.

Adding the above across i and j and observing that $Y_i^{(n)} - X_i^{(n)} + Y_j^{(n)} - X_j^{(n)} = 0$ in equilibrium gives

$$0 = r_n[\frac{1}{\varepsilon_n} \cdot \left(\frac{\delta_i \mu^{(n)}}{S^{(n)}} - 1\right)] + r_n[\frac{1}{\varepsilon_n} \cdot \left(\frac{\delta_j \mu^{(n)}}{S^{(n)}} - 1\right)]$$

By our assumption about q_n **(A2.3.2)**, $r_n(-x) = -r_n(x)$. Thus we have

$$r_n[\frac{1}{\varepsilon_n} \cdot \left(\frac{\delta_i \mu^{(n)}}{S^{(n)}} - 1\right)] = r_n[-\frac{1}{\varepsilon_n} \cdot \left(\frac{\delta_j \mu^{(n)}}{S^{(n)}} - 1\right)]$$

Since r_n is strictly increasing, we have

$$\frac{\delta_i \mu^{(n)}}{S^{(n)}} - 1 = -\frac{\delta_j \mu^{(n)}}{S^{(n)}} + 1 \Rightarrow$$

$$S^{(n)} = \frac{\delta_i + \delta_j}{2} \mu^{(n)} = \frac{\delta_i + \delta_j}{2} \cdot E^{(n)}(D_1)$$

We see that as the friction parameter ε_n disappears from the expression of $S^{(n)}$ and we obtain a risk-neutral evaluation of the price where the discount is the average of δ_i and δ_j. However, the equilibrium demand by i does depend on ε_n. In fact from (3.1) above and the expression for $S^{(n)}$, we get

$$Y_i^{(n)} - X_i^{(n)} = \frac{2}{(\delta_i + \delta_j)\mu_n} r_n[\frac{1}{\varepsilon_n}\left(\frac{2\delta_i}{\delta_i + \delta_j} - 1\right)]$$

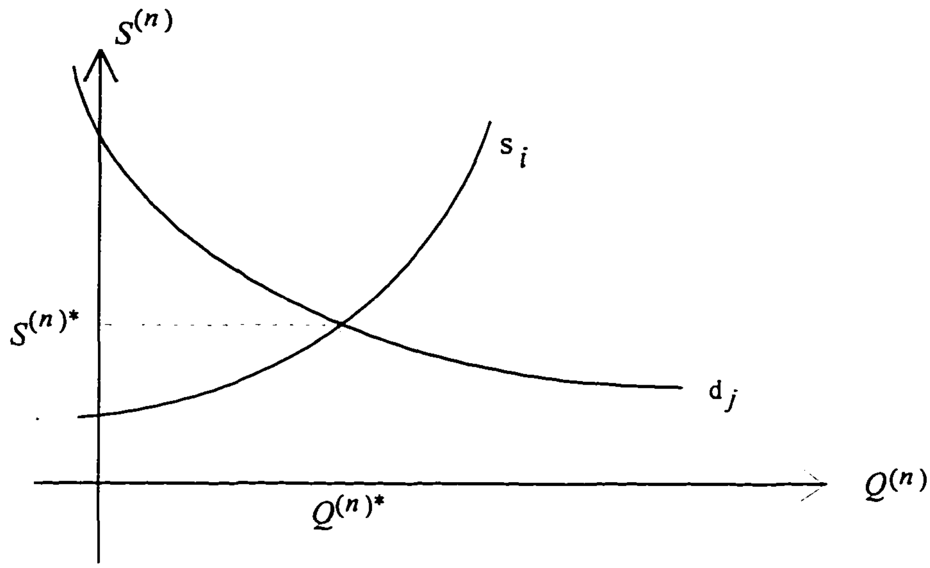
Thus as long as $\delta_i \neq \delta_j$, $Y_i^{(n)} - X_i^{(n)} \neq 0$, i.e. there is always trade. If $\delta_i < \delta_j$, then $Y_i^{(n)} - X_i^{(n)} < 0$, (notice that the space of exogenous shocks, i.e. dividends is Z which is assumed to be contained in $\mathbb{R}^N_{++}$. Hence $\mu \in \mathbb{R}^N_{++}$). Thus investor j buys and investor i sells. This is consistent with our intuition. Investor j does not discount future income as much as i does. Hence j has more incentive to buy stock at present for future consumption. It's clear that we may relax the differentiability assumption on q_n to allow kinks at $x = 0$. The above $Y_i^{(n)}$ is still optimal since **FOC** is sufficient In other words we may allow $q_n(x) = f_n(|x|)$ as long as strict convexity in x is retained.

Why is equilibrium price independent of the friction parameter ε_n ? Suppose $\delta_i < \delta_j$ so that investor i is a net supplier and j a net demander. Let's consider agent i's optimal supply and j's optimal demand as a function of $S^{(n)} > 0$. We have

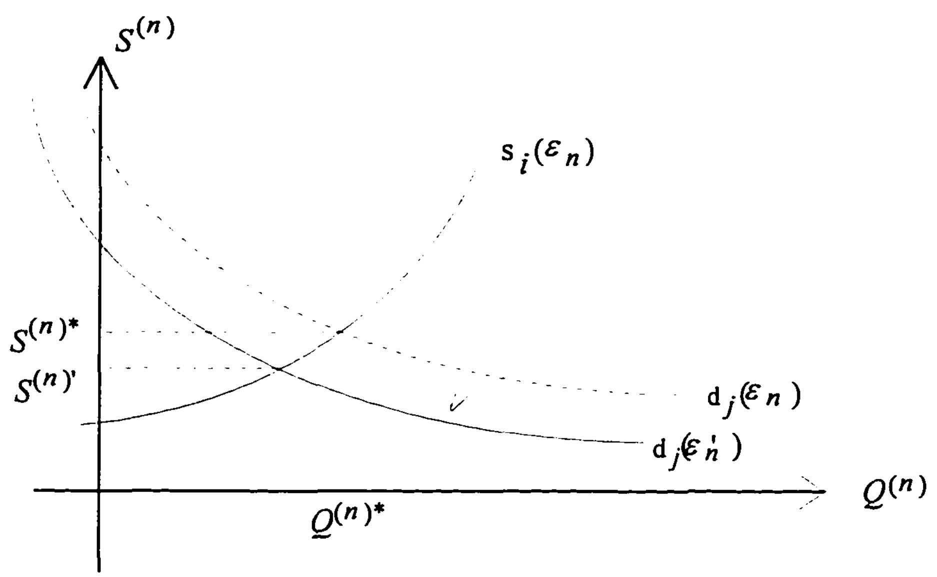
$$s_i(S^{(n)}) \equiv X_i^{(n)} - Y_i^{(n)} = -\frac{1}{S^{(n)}} \cdot r_n[\frac{1}{\varepsilon_n} \cdot \left(\frac{\delta_i \cdot \mu^{(n)}}{S^{(n)}} - 1\right)]$$

$$d_j(S^{(n)}) \equiv Y_j^{(n)} - X_j^{(n)} = \frac{1}{S^{(n)}} \cdot r_n[\frac{1}{\varepsilon_n} \cdot \left(\frac{\delta_j \cdot \mu^{(n)}}{S^{(n)}} - 1\right)]$$

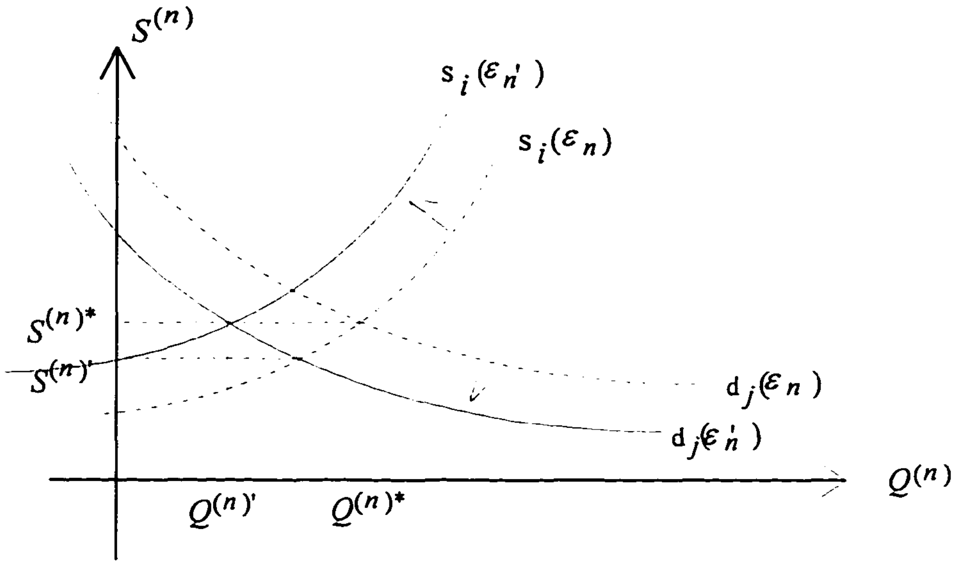
At equilibrium, $s_i(S^{(n)*}) = d_j(S^{(n)*})$ as is shown in the following diagram.



Suppose a shock occurs to the friction parameter ε_n so that it moves to a new value $\varepsilon_n' > \varepsilon_n$. Thus the demand curve shifts inwards to a new location and the equilibrium price is temporarily subjected to the pressure to move to the value $S^{(n)'} < S^{(n)*}$ as shown below



However, since costs are symmetric w.r.t purchase and sales, the above demand and supply schedules derived under risk-neutrality imply that the same shock creates a symmetric outward shifting effect on the supply curve so that the pressure on the equilibrium price to change is exactly netted out at the expense of the trade volume:



Although there is no visible effects of shocks in marginal friction on equilibrium prices, other endogenous variables do depend on ε_n. The trade volume always shrinks as costs rise since the number of shares traded is given by buyer's equilibrium demand (which equals seller's equilibrium supply)

$$Y_i^{(n)} - X_i^{(n)} = \frac{2}{\varepsilon_n(\delta_i+\delta_j)\mu^{(n)}} \cdot r_n\left[\frac{1}{\varepsilon_n}\left(\frac{2\delta_i}{\delta_i+\delta_j}-1\right)\right]$$

One might conjecture that as long as one raises the friction parameter ε_n high enough, investors will be obstructed from any trade as the payoff of any given investment (which is fixed w.r.t to increase in ε_n) will no longer be sufficient to compensate for the cost. It is true that given any positive amount of trade no matter how insignificant it is, one may always raise the parameter high enough to make it inferior to zero trade. But this is not the way investors' optimization problem works. We should take all exogenous parameter as given and then optimize. The question should then be: given any level ε_n, no matter how high it is, is there a positive amount of trade that is better than no trade? The answer is yes at least in our model. The net trade is always nonzero (as long as $\delta_i \neq \delta_j$) at any level of ε_n. The reason is simple, at any level of ε_n, the first order convexity effect approaches to zero when transaction size is arbitrarily small (since $q_n'(0) = 0$). Hence a small amount of transaction can still be optimal. Notice that the presence of friction at a given degree may actually block some individuals from trading. But notice that we should interpret an agent in our model as a group of people (with similar tastes). As individuals drop out of the market one by one, aggregate demand gradually approaches to zero (but does not have to be exactly zero).

2.4 General Case: Asymmetric structure with Multiple Agents

We now study the more general dynamic model in which there are m risk-neutral investors differentiated by their discount parameter δ_i, $i = 1, 2, \ldots, m$. The function of friction premium q_n is no longer assumed to be symmetric. In fact there is no reason to believe that costs are symmetric w.r.t purchase and sales. We may even assume that markets discriminates against individuals in that different investors may incur different market friction in a given market even if the transaction volume is the same. An anecdotal story could be that a person (or institution) with a good credit history, having a good relation with the investment banks, or belonging to certain privileged groups may incur far less search costs than a person with different backgrounds. Thus let q_n^i be the friction premium to investor i in the market for stock n. Investor i's recursive problem now becomes

$$V^i_t(X, D) = \max_{Y_i} \{X \cdot D - S_t \cdot (Y_i - X) - \Sigma_n \varepsilon_t^{(n)} \cdot q_n^i\, [S_t^{(n)} \cdot (Y_i^{(n)} - X^{(n)})] +$$
$$+ \delta_i \int_Z V_{t+1}(Y_i, \bar{D}) P(D, \mathrm{d}\bar{D}),\ 0 \le t < T-1$$

$$V^i_T(X, D) = X \cdot D$$

where $\varepsilon_t^{(n)}$ serves as the marginal friction parameter in the nth security market in the t-th trading period.

Here we assume that

A2.4.1 $q_n^i : \mathbb{R} \to \mathbb{R}_+$ is differentiable and strictly convex, with $q_n^i(0) = 0$, $\forall\, i$, $\forall\, n$.

Notice that under **A2.4.1**, the marginal cost $q_n^i{}'$ is strictly increasing and also we have $q_n^i{}'(0) = 0$ since q_n^i attains minimum at 0. Thus $q_n^i{}'$ has a inverse function called r_n^i, which is also strictly increasing with $r_n^i(0) = 0$. This observation plays an important role in our subsequent analysis.

A2.4.2 There is a constant, $N \times N$ matrix G such that the family of transition probability measures $\{P_{t,t+1}(\cdot,\cdot),\ 0 \le t \le T-1\}$satisfies,

$$\int_Z D \cdot P_{t,t+1}(\bar{D}, \mathrm{d}D) = G \cdot \bar{D}\ ,\ 1 \le t \le T,\ \forall\ \bar{D} \in Z^N$$

The above assumption simply says that the dividend process $D \equiv \{D_t\}_{0 \le t \le T}$ satisfies the growth condition $E[D_{t+1} | D_t] = GD_t$ so that by recursive substitution, we get

$$E[D_{t+m}|D_t] = G^m \cdot D_t.$$

For example, if $A = \text{diag}\{g_1, g_2, ..., g_N\}$, then we get

$$E[D_{t+m}^{(n)}|D_t] = g_n^m \cdot D_t^{(n)}$$

which says the net growth rate of the dividends of the nth security is $g_n - 1$. It is non-essential to the qualitative results but it does simplify computation later on. In the following, we denote by $g_{nm}^{(t)}$ the (n, m)-entry of G^t

Prop 2.4.1 Assume **A2.4.1** and **A2.4.2**. Suppose $\{S_0, S_1, ..., S_{T-1}\}$ is a nonzero equilibrium price process. Then in each period $T - t$ with $1 \le t \le T$, S_{T-t} satisfies

$$(2.4.1) \quad 0 = \sum_i r_n^i \left(\frac{1}{\varepsilon_{T-t}^{(n)}} \cdot \left(\frac{(\sum_{k=1}^t \delta_i^k) \sum_{m=1}^N g_{nm}^{(t)} D_{T-t}^{(m)}}{S_{T-t}^{(n)}} - 1 \right) \right), 0 \le t \le T-1, \ \forall n$$

where r_n^i is the inverse function of the marginal cost $q_n^{i\,\prime}$

proof Contained in the proof of **Theorem 2.4.4** in Appendix A.

QED

To drive home the point that we need to have structural imbalance in market friction in order to have structural effects, let's look at the following corollary.

Cor 2.4.2 Let n be fixed. If there is a number k so that each q_n^i is homogeneous of degree k, $\forall\ i$, then the cost parameter ε_n has no effects on the equilibrium price $S^{(n)}$ in each and every period.

proof Since q_n^i is homogeneous of degree k, $q_n^{i\,\prime}$ is homogeneous of degree $k-1 \geq 1$, i.e. $q_n^{i\,\prime}(\lambda \cdot x) = \lambda^{k-1} \cdot q_n^{i\,\prime}(x)$, $\forall\ x$, $\forall\ \lambda > 0$. But then r_n^i satisfies $r_n^i(\lambda^{k-1} \cdot y) = \lambda \cdot r_n^i(y)$, or

$$r_n^i(\lambda \cdot y) = \lambda^{\frac{1}{k-1}} \cdot r_n^i(y),\ \ \forall\ y, \forall\ \lambda > 0.$$

It's clear now from the equilibrium condition in **Prop 2.4.1** that the factor ε_n^{-1} can be canceled.

QED

For example if

$$q_n^i(x) = a_i|x|^k,\ k \geq 2,\ \forall\ i,$$

then no relation of shocks on ε will bear on the equilibrium prices.

The above results sharply points to the fact that when certain kinds of structural balance prevails in the market friction, then increasing or decreasing overall friction parameters will not affect the equilibrium prices. The intuitive reason is similar to the two-agent with symmetric case: under certain kind of uniformity or homogeneity, the pressure on the prices created due to shift of the aggregate demand curve of one group of agents is exactly offset by the symmetric shifts of the aggregated supply curve of the rest of the agents. In this case we say that the market friction obtains "structural balance". Therefore, in order to have visible effects of the friction parameters, the friction structure needs to be "skewed". In order to understand the working principles of structural imbalance, we first give the following.

Prop 2.4.3 Assume **A2.4.1** and **A2.4.2**. Then investor *i*'s optimal policy in the $(T-t)$-th trading period, as a function of the initial portfolio and dividend pair (X, D) as well as the vector S_{T-t} of market prices and the vector ε_{T-t} of friction parameters, is given by

$$\varepsilon_{T-t}^{(n)} q_n^{i\,\prime}\left(S_{T-t}^{(n)}[Y_{T-t}^{(n),i}(X,D)-X^{(n)}]\right)=\frac{(\sum_{k=1}^{t}\delta_i^k)\cdot\sum_{m=1}^{N}g_{nm}^{(t)}\cdot D_{T-t}^{(m)}}{S_{T-t}^{(n)}}-1,$$

$$0\le t\le T-1,\quad \forall n \qquad (2.4.2)$$

or equivalently,

$$Y^{i(n)}_{T-t}(X,D)-X^{(n)}=\frac{1}{S_{T-t}^{(n)}}r_n^i\left(\frac{1}{\varepsilon_{T-t}^{(n)}}\cdot\left(\frac{(\sum_{k=1}^{t}\delta_i^k)\sum_{m=1}^{N}g_{nm}^{(t)}D_{T-t}^{(m)}}{S_{T-t}^{(n)}}-1\right)\right)$$

$$0\le t\le T-1,\quad \forall n \qquad (2.4.3)$$

proof Contained in the proof of **Theorem 2.4.4** in Appendix A.

QED

Notice that $Y_{T-t}^{(n),i}(X, D) - X^{(n)}$ is agent i's order (number of shares) in the nth security market in period $T - t$. When $Y_{T-t}^{(n),i}(X, D) - X^{(n)} > 0$, agent i is a net buyer, if $Y_{T-t}^{(n),i}(X, D) - X^{(n)} < 0$, agent i is a net seller. Therefore the transaction size (in shares) for agent i in the nth security market in period $T - t$, is defined to be

$$|Y_{T-t}^{(n),i}(X, D) - X^{(n)}|.$$

Equation (2.4.2) is simply the classical optimality condition, namely, marginal revenue (the RHS) must be equal to marginal cost (the LHS) (see also footnote 1). To gain some insight into structural imbalance, let's look at the case of two agents i and j. In equilibrium, suppose i is a buyer and j the seller and since demand is equal to supply we must have

$$Y_{T-t}^{(n),i}(X, D) - X^{(n)} = -[Y_{T-t}^{(n),j}(X, D) - X^{(n)}],$$

or equivalently,

$$|Y_{T-t}^{(n),i}(X, D) - X^{(n)}| = |Y_{T-t}^{(n),j}(X, D) - X^{(n)}|$$

When the market experiences a shock in marginal friction, say, the marginal friction parameter $\varepsilon_{T-t}^{(n)}$ reduces to a lower level, then both agents have incentives to increase their transaction sizes since marginal revenue is higher than marginal cost for the buyer i and marginal revenue is lower than marginal cost for seller j (notice that for seller j, both the marginal revenue and the marginal cost are negative). This is also seen mathematically since the marginal friction function $q_n'(x)$ is increasing in x.

Therefore, the magnitude of increase in transaction size in buyer i's readjustment of optimal policy depends on how fast marginal friction $q_n'(x)$ (>0) increases w.r.t to an increase in x (>0). Seller j's analysis mirrors the buyer's. The magnitude of increase in transaction size by seller j depends on how fast marginal friction $q_n'(x)$ (< 0) decreases w.r.t to a decrease in x (< 0). If $q_n'(x)$ increases slower on the buyer side (i.e. w.r.t an increase in $x > 0$) than it decreases on the seller side (i.e. w.r.t a decrease in $x < 0$), then, to obtain optimality, buyer i will have to increase his/her demand more than seller j increases his/her supply. In this case, the nth security is under-supplied in the $(T-t)$th period and the market price $S_{T-t}^{(n)}$ will have to adjust upwardly to get back in equilibrium. This is the case which I refer to as "structural bias in buyers' favor", i.e. the overall market friction works systematically in favor of buying activities in that the marginal cost born by the buyer increases at a slower rate than by the seller when both sides try to expand their trade. In my model, this is precisely the reason why market will attract more buying activities causing a demand-driven price hike when the friction parameter reduces.

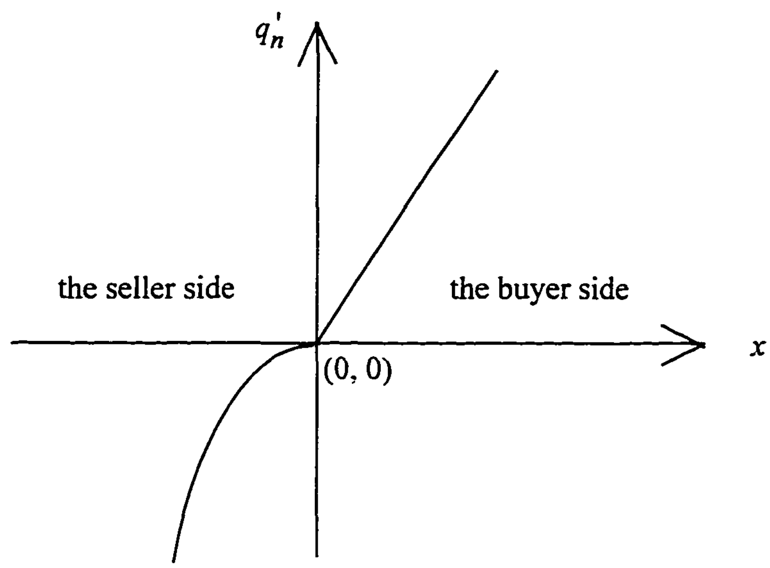
Based upon this intuitive understanding of structural imbalance, it is not surprising to see that the following hypothetical function for friction premium exhibits structural bias toward the buyer in the case of a two-agent model.

(2.4.4)
$$q_n(x) = \begin{cases} x^2 & \text{if } x \geq 0; \\ -x^3, & \text{if } x \leq 0 \end{cases}$$

The marginal friction is given by

$$q_n'(x) = \begin{cases} 2x & \text{if } x \geq 0; \\ -3x^2, & \text{if } x \leq 0 \end{cases}$$

which is illustrated by the following graph:



Clearly, the market friction takes a heavier toll on the seller side when both sides try (unilaterally) to expand transaction sizes since the signed marginal friction $q_n'(x)$ decreases (i.e. increases in absolute value) much faster as x (<0) becomes more negative (i.e. increases in absolute value). We shall later on prove that in the above example, the price of the nth security increases (along with it's volatility) as the nth market improves in overall efficiency.

We now try to give a general characterization of structural imbalance of market friction. For this purpose, we need to solve out the above equilibrium.

Fortunately, the dynamic equilibrium model admits a closed form solution as given in the following theorem. To keep track of notation, we use i, j to denote agents, n to denote the nth security, and t to denote time periods. r_n^i is the inverse of marginal friction $q_n^{i}{}'$ to agent i in the nth stock market and $\varepsilon_t^{(n)}$ is the marginal friction parameter in the nth market and the tth period. δ^i is agent i's time-preference parameter, and finally, $g_{nm}^{(t)}$ is the (n, m)-entry of G^t, where G is the growth matrix mentioned in **A2.4.2**.

Theorem 2.4.4 Under **A2.4.1** and **A2.4.2**, the model admits a unique nonzero equilibrium prices process $S = \{S_t\}_{0 \le t \le T-1}$. S is independent of the initial condition, i.e. the initial portfolio distribution X_0 among agents as well as the initial value of dividend D_0. Moreover, in each period t with $1 \le t \le T$, S_{T-t}, which depends only on the current value D_{T-t} observed, is strictly positive and has the form

$$S_{T-t}^{(n)}(D_{T-t}) = \frac{1}{w_{T-t}^{(n)}} \cdot \sum_{m=1}^{N} g_{nm}^{(t)} D_{T-t}^{(m)}, \quad 1 \le t \le T,\ 1 \le n \le N, \tag{2.4.5}$$

where $w_{T-t}^{(n)}$ is the unique non-zero number satisfying the equation

$$\sum_{i=1}^{m} r_n^i \left(\frac{1}{\varepsilon_{T-t}^{(n)}} \cdot \left(w_{T-t}^{(n)} \cdot \sum_{k=1}^{t} \delta_i^k - 1 \right) \right) = 0 \tag{2.4.6}$$

Moreover, the optimal portfolio policy by i in each period is given by

$$Y^{i(n)}_{T-t}(X,D) - X^{(n)} = \frac{w^{(n)}_{T-t}}{\sum_{m=1}^{N} g^{(t)}_{nm} D^{(m)}_{T-t}} \cdot r^{i}_{n}\left(\frac{1}{\varepsilon^{(n)}_{T-t}} \cdot \left(w^{(n)}_{T-t} \sum_{k=1}^{t} \delta^{k}_{i} - 1\right)\right)$$

$1 \le t \le T, \quad \forall n$ (2.4.7)

and the value function of i is given by

$$V^{i}_{T-t}(X,D) = \left(\sum_{k=0}^{t} \delta^{k}_{i}\right)(G^{t}D)^{\mathsf{T}} \cdot X + f^{i}_{T-t}, \quad 1 \le t \le T \tag{2.4.8}$$

where f^{i}_{T-t} is recursively determined through the following:

$$f^{i}_{T-t} = \sum_{n} \frac{1}{\varepsilon_{n}} \left(w^{(n)}_{T-t} \cdot \sum_{k=1}^{t} \delta^{k}_{i} - 1\right) \cdot r^{i}_{n}\left(\frac{1}{\varepsilon_{n}} \cdot \left(w^{(n)}_{T-t} \cdot \sum_{k=1}^{t} \delta^{k}_{i} - 1\right)\right)$$

$$- \sum_{n} q^{i}_{n} \circ r^{i}_{n}\left(\frac{1}{\varepsilon_{n}} \cdot \left(w^{(n)}_{T-t} \cdot \sum_{k=1}^{t} \delta^{k}_{i} - 1\right)\right) + \delta_{i} \cdot f^{i}_{T-t+1}, \quad 1 \le t \le T,$$

$$f^{i}_{T} = 0 \tag{2.4.9}$$

proof Backward induction and application of **Lemma A.1** (in the Appendix A) in each iteration step. See Appendix A.

QED

As is seen, the price risks of a security are entirely driven by risks of the underlying project reflected in corresponding dividend process. The market mechanism contributes to the price risks only by multiplying the stochastic dividend (in a given period) by a constant factor $1/w^{(n)}_{T-t}$. It is nice to see that all friction effect on equilibrium prices are contained in the coefficient $w^{(n)}_{T-t}$. Call the function $w^{(n)}_{T-t}$ the distortion function in market n and period $T - t$. An increase (or decrease) in $w^{(n)}_{T-t}$ brings about a decrease (or increase) in the equilibrium price $S^{(n)}_{T-t}$ as well as its volatility. Hence, the study of friction effects hangs critically on the behavior of the distortion functions.

To simplify notation, let's consider a change of variables given by

$$\theta^{(n)}_t = \frac{1}{\varepsilon^{(n)}_t}$$

for each security n and each trading period t. $\theta^{(n)}_t$ may be regarded as a parameter representing market efficiency. Hence a reduction in $\varepsilon^{(n)}_t$ is interpreted as an improvement in market efficiency. Then (4.6) becomes

$$\Sigma_i r^i_n [\theta^{(n)}_{T-t} \cdot (\alpha_{(T-t),i} \cdot w^{(n)}_{T-t} - 1)] = 0 \qquad (2.4.10)$$

where $\alpha_{(T-t),i} = \Sigma_{k=1}^{t} \delta^k_i$. Equation (4.10) determines $w^{(n)}_{T-t}$ as a function of $\theta^{(n)}_{T-t}$ with $\theta^{(n)}_{T-t}$ ranging all over $\mathbb{R}_{++}$. Note well that $w^{(n)}_{T-t}(\theta^{(n)}_{T-t}) > 0$ in each period $T - t$, and (4.10) implies that there is some value $a > 0$ s.t. for all δ_i with $\delta_i < a$, $\theta(\alpha^i_{T-t} \cdot w^{(n)}_{T-t} - 1) < 0$ and for all those i's with $\delta_i \geq a$, $\theta(\alpha^i_{T-t} \cdot w^{(n)}_{T-t} - 1) > 0$. In other words, for those i with $\delta_i < a$, investor i is a net seller in the nth market and for all those i with $\delta_i \geq a$, investor i is the net buyer in the nth market. Therefore if $\delta_1 < \delta_2 < \ldots < \delta_m$, then investor 1 is always a net seller and investor m is always a net buyer in every market and in every period.

As seen from the previous discussion, when the market friction leans in favor of one side of a transaction, this side has more freedom in adjusting its aggregate demand (or supply) than the other side since the marginal friction changes at a slower rate. This is the essence of structural imbalance. This can also be captured in terms of the inverse function of marginal friction. A slower changing marginal friction $y = q_n'(x)$ at $x > 0$ (or $x < 0$) is equivalent to a faster changing inverse $r_n(y)$ at $y > 0$ (or $y < 0$). In fact, equation (2.4.3) in **Prop 2.4.3** directly ties the individual order size to the inverse of marginal friction and it clearly demonstrates that individual who faces a slower changing marginal friction will respond more violently in adjustment of his (her) transaction size, be he (she) a seller or a buyer. The following is my formal definition of structural imbalance.

DEF Let r^i_n be the inverse of marginal cost functions $q^i_n{}'$, $i = 1, 2, \ldots, m$. Suppose they satisfy the following condition: for every $\theta > 0$, and for every vector of real numbers $(a_1, a_2, \ldots, a_m) \neq 0$ with

$$\Sigma_i\, r^i_n(\theta a_i) = 0 \qquad (2.4.11)$$

there exists a number $\rho > 0$ such that for every θ in the interval $(\theta,\ \theta + \rho)$,

$$\Sigma_i\, r^i_n(\theta \cdot a_i) > 0 \qquad (2.4.12)$$

Then we say that the nth security market exhibits structural bias in favor of the buyers. If the relation " >" in (2.4.12) is replaced with " < ", then we say the nth security market exhibits structural bias in favor of the sellers. If it is replaced with an equal sign " = ", then we say the *n*th market obtains structural balance.

To understand the definition, let's first look at (2.4.11). We may regard θ as the reciprocal of the friction parameter, i.e. θ is the efficiency parameter, and let

$$a_i = \frac{D_{T-t}^{(n)} \sum_{k=1}^{t} \delta_i^k}{S_{T-t}^{(n)}} - 1$$

Then, from (2.4.3), $r_n^i(\theta a_i)$ is simply agent *i*'s demand or supply (modulo a common coefficient). Thus (2.4.11) is simply the equilibrium condition: aggregate supply is equal to aggregate demand, with the positive terms representing individual demands and the negative terms representing individual supplies. Suppose the equilibrium price $S_{T-t}^{(n)}$ is rigid at first (that's why a_i is a constant). When θ is increased to a higher level θ', every agent wants to unilaterally increase his (her) trade size so the positive terms in (2.4.11) will become more positive and the negative terms in (2.4.11) more negative. But (2.4.12) says that the net result is that aggregate demand exceeds aggregate supply (before price adjustment). In other words, at an aggregate level, buyers increase their demand more than sellers increase their supply because the marginal friction function changes at a slower pace on the buyer side than on the sell side. This is in term reflected in faster changes in the inverse function of the marginal friction on the buyer side than on the seller side as in (2.4.12). In this case, pressure is on the price to increase to a higher level to put the market back in equilibrium, which is shown to be indeed the case in the following theorem.

Lemma 2.4.6 Assume **A2.4.1** and **A2.4.2**. For every n, if the nth security market exhibits structural bias in favor of the buyers (respectively, sellers), then in each trading period, the distortion function $w_{T-t}^{(n)}$ is a strictly increasing (respectively, decreasing) function of the efficiency parameter $\theta_{T-t}^{(n)}$. If the nth security market obtains structural balance, then $w_{T-t}^{(n)}$ is constant w.r.t. $\theta_{T-t}^{(n)}$.

proof see Appendix A.

QED

Theorem 2.4.7 Assume **A2.4.1** and **A2.4.2**. For every n, if the nth security market exhibits structural bias in favor of the buyers (respectively, sellers), then in each trading period, a reduction in the friction parameter $\varepsilon_{T-t}^{(n)}$ (equivalently, an increase in $\theta_{T-t}^{(n)}$) will drive the equilibrium price $S_{T-t}^{(n)}$ up (respectively, down) along with its volatility (measured by price variance). If the nth security market obtains structural balance, then the equilibrium price $S_{T-t}^{(n)}$ (and its volatility) is constant w.r.t. changes in the friction parameter.

proof Direct consequence of equation (2.4.5) and the above lemma.

QED

In my model, the volatility of asset price increases (or decreases) because the price goes up (or down) by the same constant scaling factor λ in every contingent state of nature. As a result, the mean price goes up (or down) by a factor of λ and the variance goes up (or down) by a factor of λ^2. Thus, in the case of structural bias in favor of the buyers, the rise of price volatility is completely due to an increase in the magnifying factor of volatility of the underlying risky projects. This type of volatility increase is intimately tied with structural imbalance of market friction (which contributes to the increase in the distortion factor $w_{T-t}^{(n)}$) as well as the project risks and therefore can not be abated by reducing noise making trading activities. Thus my model suggest that, unless the structure of market friction is changed, we'd better prepare for even higher volatility in security prices as the markets continue to improve in overall efficiency.

Apart from its economic significance, we now demonstrate the power of the above theorem in determining to which side the friction structure is tilted and hence how price and volatility will react to changes in the friction parameters. Recall the example in (4.4) in which the marginal friction function is given by

$$q_n(x) = \begin{cases} 2x, & \text{if } x \geq 0; \\ -3x^2, & \text{if } x \leq 0 \end{cases}$$

Hence the inverse function is given by

$$r_n(y) = \begin{cases} y/2, & \text{if } y \geq 0; \\ -\sqrt{-y/3}, & \text{if } y \leq 0 \end{cases}$$

Now let a, b be two nonzero numbers and $\theta > 0$ s.t.

$$r_n(\theta a) + r_n(\theta b) = 0$$

Then we must have that either a or b is negative and the other is positive. WOLG suppose that $a < 0$ and $b > 0$. Then

$$\theta b/2 - \sqrt{-\theta a/3} = 0 \Rightarrow \theta^2 b^2/4 = -\theta a/3 \Rightarrow \theta b^2/4 = -a/3.$$

If θ is increased to a higher level $\theta' > 0$, then

$$\theta' b/4 > -a/3b \Rightarrow \theta'^2 b^2/4 > -\theta' a/3 \Rightarrow \theta' \cdot b/2 > \sqrt{-\theta' a/3} \Rightarrow$$

$$\Rightarrow \theta' \cdot b/2 - \sqrt{-\theta' a/3} > 0 \Rightarrow r_n(\theta' \cdot a) + r_n(\theta' \cdot b) > 0$$

Therefore, equilibrium price $S^{(n)}$ reacts positively to an increase in θ_n.

Finally, it is almost a tautology that market friction always works against trade. This is also revealed by the fact that an increase in the friction parameter will always cause a decrease in the equilibrium trade volume.

Theorem 2.4.8 Assume **A2.4.1** and **A2.4.2**. Then in any case, the equilibrium trade volume is always strictly decreasing in the friction parameters in each and every trading period and every security market.

proof See Appendix A.

QED

It follows that structural bias in favor of buyers in market friction will result in the increase of security prices, volatility and trade volumes in each trading period when markets improve in efficiency in that period. Since the (expected) return of security n in period $T-t$ is given by

$$R_{T-t}^{(n)} = \frac{E[S_{T-t+1}^{(n)} + D_{T-t+1}^{(n)} | D_{T-t}]}{S_{T-t}^{(n)}}$$

the structural bias in buyers' favor together with a rapid improvement in overall efficiency in the nth security market from period $T-t+1$ on will inevitably result in an increase in the $R_{T-t}^{(n)}$ and may actually help explain and predict a growing trend in equity returns and thus may even suggest that a steady improvement in market efficiency from period to period will actually exacerbate the excessive equity premium anomaly over time instead of alleviating it.

2.5 Structural Imbalance vs. Risk-aversion

By now we are ready to see why cost effects in general can be conceptually decomposed into two parts, namely effects through structural imbalance and effects through agent risk aversion. If agents are risk-neutral, then any cost effects on equilibrium prices must be realized through inherent cost imbalance. In other words, if the friction structure is such that for every $\theta > 0$, and for every vector of real numbers $(y_1, y_2, ..., y_m)$ (m is the number of agents)

$$0 = \Sigma_i r_n^{\prime i}(\theta y_i) \Rightarrow 0 = \Sigma_i r_n^{\prime i}(y_i) \qquad (2.5.1)$$

then no effect is left when agents are risk neutral. Here r^i_n is the inverse function of marginal friction $q^i_n{}'$ facing agent i in the nth stock market. Special cases of structural homogeneity are when the friction premium function is symmetric or when the function is homogeneous of degree k. If cost obtains structural homogeneity, then it can only impact asset prices through agent risk aversion in that it may further distort the risk-premium in the stock prices as is indicted by the following two-period two - person static model. Let the period 1 security payoff matrix D_1 be multivariate normally distributed: $D_1 \sim N(\mu, \Sigma)$. Let investor i has CARA *Von-Neumann Morgenstern* utility function $u^i(w_0, w_1)$ over period-0 wealth w_0 and period-1 wealth w_1 given by

$$u^i(w_0, w_1) = -\exp[-R_i(w_0 + w_1)],$$

where $R_i > 0$ is agent i's *Arrow-Pratt* measure of risk aversion. Notice that

$$w_0 = X_i \cdot D_0 - S \cdot (Y_i - X_i) - \Sigma_n q_n[\varepsilon_n S_n \cdot (Y_{i\,n} - X_{i\,n})],$$

while $w_1 = Y_i \cdot D_1$ is normally distributed: $w_1 \sim N(Y_i^{\mathrm{T}}\mu,\ Y_i^{\mathrm{T}}\Sigma Y)$. Thus, $E[u^i(w_0, w_1)]$ is actually of mean-variance type given by

$$E[u^i(w_0, w_1)] = -\exp[-(R_i \cdot w_0 + R_i \cdot Y_i \cdot \mu - \frac{1}{2} R_i^2 \cdot Y_i^{\mathrm{T}} \Sigma Y_i)]$$

In other words, agent i's problem is equivalently given by

$$\max_{Y_i} \{X_i \cdot D_0 - S \cdot (Y_i - X_i) - \Sigma_n q_n[\varepsilon_n S_n \cdot (Y_{i\,n} - X_{i\,n})] + Y_i \cdot \mu - \frac{1}{2} R_i \cdot Y_i^{\mathrm{T}} \Sigma Y_i)\}$$

To simplify computation, we assume that $\Sigma = \text{diag}\{\sigma_1^2, \ldots, \sigma_N^2\}$, i.e. the security payoffs are uncorrelated.

FOC:

$$Y_{in}:\ -S_n - \varepsilon_n \cdot S_n \cdot q_n'[\varepsilon_n S_n \cdot (Y_{in} - X_{in})] + \mu_n - R_i \sigma_n^2 \cdot Y_i = 0,\ \forall\, n \Rightarrow$$

$$r_n\left(\frac{1}{\varepsilon_n}\left(\frac{1}{S^{(n)}}(\mu_n - \sigma_n^2 R_i Y_{in}) - 1\right)\right) = \varepsilon_n \cdot (Y_{in} - X_{in}) \cdot S_n$$

Adding across i, j (two persons) and assuming symmetric cost structure, we get

$$\frac{1}{S^{(n)}}(\mu_n - \sigma_n^2 R_i Y_{in}) + \frac{1}{S^{(n)}}(\mu_n - \sigma_n^2 R_j Y_{jn}) = 2, \text{ or}$$

$$S^{(n)} = \mu_n - \frac{1}{2}\sigma_n^2(R_i^2 \cdot Y_{in} + R_j^2 \cdot Y_{jn}) \qquad (*)$$

From (*) we observe the following

:

(i) The equilibrium price $S^{(n)}$ is decomposed into two parts: the first part is just the expected payoff of stock n (the risk-free return is zero both agents treat period 0 and period 1 consumption equally). The second part $\frac{1}{2}\sigma_n^2(R_i Y_i^{(n)} + R_j Y_j^{(n)})$ is the equilibrium risk premium. Indeed, if the nth security happens to be risk-free, then $\sigma_n = 0$ and we have

$$S^{(n)} = \mu_n$$

(ii) Since the structural effect of the cost is nonexistent, the first part, i.e. the discounted (according to risk-free rate) expected payoff μ_n is not affected by cost parameter. On the other hand, the risk premium

$$\frac{1}{2}\sigma_n^2(R_iY_i^{(n)} + R_jY_j^{(n)})$$

contains the effect of the cost parameter ε_n which is implicit in the equilibrium demand $Y_i^{(n)}$ and $Y_j^{(n)}$ (note that equilibrium demand $Y_i^{(n)}$ is a function of R_i, ε_n and $S^{(n)}$ through the **FOC**). Thus we see that when no structural effect exists, the impact of cost is realized only through agents' risk aversion in that it may distort the equilibrium risk premium

$$\frac{1}{2}\sigma_n^2(R_iY_i^{(n)} + R_jY_j^{(n)}).$$

Most of the existing literature that examines proportional types (including convexified version) of cost effects on asset prices (or returns) have assumed symmetric cost functions and have thus excluded the structural effects of costs which may actually be of first order degree importance when risk aversion is negligible.

2.6 Conclusion

Price and trade volume are the two most readily observable quantities in the financial market and an analysis of how the two variables react to shocks in exogenous variables is of foremost importance for a better understanding of the working relations among variables in a financial economy in general. In this paper I study how stock prices and trade volumes react to changes in the friction parameter $\varepsilon_t^{(n)}$ in each security market n and trading period t in an attempt to better understand how market friction may contribute to price distortions at a somewhat more fundamental level. I observe that the effects on prices can be conceptually decomposed into two parts, namely, effects through structural imbalance and effects through risk-aversion. I remove risk aversion from our model and examine purely the first type effects. I find that price of stock n is a linear function of its fundamental value and the first type effects are entirely reflected in the distortion function which contains no stochastic element and acts as a coefficient in the linear relation. We find that in general, if the friction structure is biased in buyers' favor in the sense that buyers may have more freedom in adjusting their aggregate demand because of a slower growth rate of marginal friction than the sellers in the face of a reduction of the friction parameter in the nth security market, then equilibrium price $S^{(n)}$ (and its volatility) rises in every period, the reason being that the demand side puts more upward pressure on the price than can be offset by the downward pressure from the supply side. The effects are exactly the opposite if the structure is biased otherwise. In any case, equilibrium trade volume always increase when marginal friction reduces because both of the two forces acts unambiguously to push the volumes up. The price risks in my model are inherently tied with the underlying industrial projects as well as market friction structure and cannot therefore be removed by simply reducing noise trading activities. We feel that the structural bias in buyers' favor together with a rapid evolution toward better market efficiency may fit more realistically with the on-going changes experienced by the asset markets. On the other hand, the structural asymmetry is less severe for the liquid bond market. If this is indeed the case, the gap between equity and bond returns will widen as market continues to evolve. Therefore it may actually help predict a growing trend of equity premium and equity price volatility. In any case, trade volume will continue to grow, which is another empirically observed phenomena (i.e. positive correlation between price volatility and trade volume).

3 Efficient Capital Markets under General Square-integrable Martingales

3.1 The General Setup of the Financial Markets

Ever since *Luis Bachelier* proposed *Brownian* motion processes as asset prices, the continuous-time models of asset pricing and derivatives valuation of late have been set up in the *Brownian* motion framework in the sense that the basic market risks of asset prices are driven by *Brownian* motion type of white noise processes. A classical example is that equity prices are assumed to be geometric *Brownian* motions. Needless to say, there are some technical advantages to the *Brownian* motion framework. First, we know that the markets are dynamically complete as long as it contains stocks and one bond (apart from some additional purely technical assumptions). Secondly, there exists an equivalent probability measure (also referred to as the risk-neutral probability) Q, s.t. the joint price process becomes a martingale under Q when discounted by the bond price process. The importance of the first property is quite obvious: any contingent claims can be replicated via stocks and bonds in the primitive asset markets that contain only stocks and bonds and therefore, in the absence of arbitrage, the prices of the claims should be equal to the market values of the replicating portfolios. The advantage of the second property is mainly computational; it allows for risk-neutral valuations of all contingent claims.

While there are some good economic justifications for the *Brownian* motion framework, it is still not clear that the prevailing structure of market risks are actually *Brownian* motion driven. As *LeRoy* pointed out in his paper (1989), in a discrete-time model, while random walk type of asset prices do imply that the asset markets are efficient, the converse is not true. In fact, for asset markets to be efficient (or arbitrage-free in modern finance parlance), it is necessary and sufficient for the joint asset price process to be a martingale under some (probably artificial) probability measure and some discount processes (often referred to as pricing kernel or pricing density). Since *Brownian* motion is the continuous-time limit of random walk processes, it is not hard to see that the classical *Brownian* motion framework of capital markets may be unnecessarily restrictive and may thus exclude many more cases in which markets are still efficient but are nonetheless not driven by *Brownian* motion type of white noise. In this paper, we attempt to replace the underlying *Brownian* motion driven market risks with general square-integrable martingales. We find that the two most important properties mentioned in the above paragraph are still retained. We then discuss the consequences of these properties in terms of arbitrage pricing of derivative securities.

We begin with a stochastic base $(\Omega, \mathcal{F}, P, \mathrm{T}, \mathrm{F})$ in which the triplet $(\Omega, \mathcal{F}, P)$ is a probability space, $\mathrm{T} = [0, T]$ is the finite time horizon to be considered, and $\mathrm{F} = \{\mathcal{F}_t\}_{t\in\mathrm{T}}$ is the filtration satisfying the usual conditions, namely, right-continuity and completeness. We assume that there are N stocks and one bond in the financial markets. We represent the price process of the nth stock by $S^{(n)} = \{S_t^{(n)}\}_{t\in\mathrm{T}}$, an F-adapted semimartingale process, with more restrictions to be imposed later on. We use $B = \{B_t\}_{t\in\mathrm{T}}$ to represent the bond price. A trading strategy is a pair (φ, θ) of predictable process, where $\varphi = \{\varphi_t\}_{t\in\mathrm{T}}$ is an $\mathbb{R}$-valued process representing bond positions and θ is an $\mathbb{R}^N$-valued process representing joint positions in equity holdings. Let $\hat{S}$ be the discounted (through bond price) equity price process. In the *Brownian* motion treatment, we know that the space of equity trading strategies θ s.t. the (discounted) process $\int \theta_u d\hat{S}_u$ of capital gains is exactly a martingale under the equivalent martingale measure Q plays an important role in eliminating arbitrage and in the valuation of securities. Now in our generalized version, we want to characterize the space of θ s.t $\int \theta_u d\hat{S}_u$ is exactly a Q-martingale, for some probability measure Q. We need some facts about stochastic integration.

Let N be a continuous local F-martingale. We denote by $\mathcal{L}^2(\langle N \rangle)$ the space of all predictable F-processes θ s.t.

$$\int_0^T |\theta_u|^2 \, d\langle N \rangle_u < +\infty, \text{ a.s.}$$

Now for every $\theta \in \mathcal{L}^2(\langle N \rangle)$, we know that $\int \theta_u dN_u$ is a square-integrable martingale iff $\theta \in \mathcal{H}^2(\langle N \rangle)$, i.e. iff θ satisfies

$$E\int_0^T |\theta_u|^2 \, d\langle N \rangle_u < +\infty$$

This is because a local martingale L is a square-integrable martingale iff $E(\langle L \rangle_T) < +\infty$. It turns out that we can also characterize in a similar way the subspace of $\mathcal{L}^2(N)$ consisting of all predictable processes θ s.t. $\int \theta_u dN_u$ is exactly a martingale, thanks to the famous *Davis* inequality.

<u>*Davis Inequality*</u>: Let $(\Omega, \mathcal{F}, P, \mathrm{T}, \mathrm{F})$ be a stochastic base satisfying the usual conditions. Let $m > 0$. Then $\exists$ real constants $a_m > 0$ and $b_m > 0$ (depending only on m) s.t. for every continuous local F-martingale N, we have

$$a_m \cdot E(\langle N \rangle_T^m) \le E[(N_T^*)^{2m}] \le b_m \cdot E(\langle N \rangle_T^m),$$

where $N^* = \sup_{0 \le t \le T} |N_t|$.

proof See *Karatzas* and *Shreve* (Pg 166, **3.28**).

QED

Fact 3.1.1 Let N be a local F-martingale. Let $m \ge 1$. Then for every $\theta \in \mathcal{L}^2(\langle N \rangle)$, $\int \theta_u \mathrm{d}N_u$ is a an L^m-integrable martingale iff

$$E\left[\int_0^T |\theta_u|^2 \, \mathrm{d}\langle N \rangle_u\right]^{m/2} < +\infty \qquad (3.1.1)$$

proof Suppose (3.1.1) is satisfied. Let $L = \int \theta_u \mathrm{d}N_u$. Then by the *Davis* inequality,

$$E[(L_T^*)^m] \le b_m \cdot E(\langle L \rangle_T^{m/2}) < +\infty \Rightarrow$$

L is L^m-integrable and is of class LD and is thus an L^m-integrable martingale. Conversely, suppose L is an L^m-integrable martingale. Then again by the *Davis* inequality, $E(\langle L \rangle_T^{m/2}) \le a_m^{-1} \cdot E[(L_T^*)^m] = a_m^{-1} \cdot E[|L_T|^m] < +\infty$.

QED

Now back in our financial economy, we impose the following assumptions on the stochastic base $(\Omega, \mathcal{F}, P, \mathrm{F})$ as a maintained hypothesis throughout this paper.

A3.1.1 The information is continuously revealed in that F is continuous, i.e. every *RCLL* F-martingale (hence every *RCLL* local F-martingale) has a continuous modification (and is therefore actually continuous itself).

A3.1.2 Let $\mathcal{M}_0^2$ be the space of all continuous and square-integrable $\mathbb{R}$-valued F-martingales N with $N_0 = 0$ a.s.. There exist finitely many members $M^{(1)}$, ..., $M^{(K)}$ of $\mathcal{M}_0^2$ s.t. $\langle M^{(i)}, M^{(j)} \rangle = 0$ for all $i \neq j$, and if N is any member of $\mathcal{M}_0^2$ with $\langle N, M^{(i)} \rangle = 0$ for all i, then $N = 0$ a.s.. Moreover, there is a nonnegative and nondecreasing continuous F-process A s.t. for every j, $\langle M^{(j)} \rangle$ is absolutely continuous w.r.t A in that there is some nonnegative, adapted and measurable process $\psi^{(j)}$ with

$$\langle M^{(j)} \rangle = \int \psi^{(j)} \mathrm{d}A, \forall j$$

Thus under **A3.1.2**, there exist finitely many members $M^{(1)}$, ..., $M^{(K)}$ of $\mathcal{M}_0^2$ that are pairwise uncorrelated in that $\langle M^{(i)}, M^{(j)} \rangle = 0$ (i.e. $M^{(i)} \| M^{(j)}$) for all $i \neq j$, and also are exhaustive in that every N in $\mathcal{M}_0^2$ that are uncorrelated with all $M^{(j)}$'s is zero. Let $M = (M^{(1)}, ..., M^{(K)})^{\mathsf{T}}$. Then M is called the martingale generator. We have the following fact called the representation theorem.

Fact 3.1.2 Assume **A3.1.1** and **A3.1.2** with M the martingale generator. Then for every local F-martingale N, there is some $\theta \in \mathcal{L}^2(\langle M \rangle)$ s.t.

$$N_t = N_0 + \int_0^t \theta_u \mathrm{d}M_u \text{ a.s.} \forall t.$$

proof See *Philip Protter*.

QED

For our purposes, we will regard M as the source of market risks and will call M the market generator. They may be considered as the continuous-time counterpart of the market factors in the classical APT of *Ross*. All market risks are generated by M and therefore we assume that the joint stock price process S is given by

$$dS = a_S \cdot \psi \cdot dA + b_S \cdot dM$$

The stocks may pay intermediate dividends that are given by a joint F-process D. We assume that

$$dD = a_D \cdot \psi \cdot dA + b_D \cdot dM$$

for some $N \times K$ matrix-valued adapted and measurable processes a_S, a_D s.t. $a_S \cdot \psi$ and $a_D \cdot \psi$ are both in $\mathcal{L}(A)$, and for some $N \times K$ matrix-valued process b_S, b_D in $\mathcal{L}^2_{N \times K}(M)$. Notice that the price S_T is the last date lump-sum dividend payoff so we do not in general have $S_T = 0$. On the other hand, D is the intermediate cumulative dividend payoff (also called the running reward of the stocks). Let $G = S + D$ be the gain process so that

$$dG = a \cdot \psi \cdot dA + b \cdot dM$$

where $a = a_S + a_D$ and $b = b_S + b_D$. Security $n = 0$ is a non-coupon paying bond with price process B given by

$$dB = \beta \cdot \psi \cdot dA$$

for some $\Re$-valued adapted and measurable process β s.t.. $\beta\cdot\psi \in \mathcal{L}(A)$. (Again, more appropriately B_T should be regarded as the last-moment dividend payoff so B is actually a gain process as usual).

A3.1.3 B is strictly positive and bounded away from zero.

We will use an *RCLL* F-process $c = \{c_t\}_{t\in\mathrm{T}}$ to denote a *cumulative cash flow* (*CCF*) process. We require that $c_0 = 0$. Thus c does not include initial investment (cash outflow) and thus represents contingent claims to future cash inflows. An investor may adopt a trading strategy (φ, θ) to finance c (presumably for consumption) with some initial investment w. Thus we say that a *CCF* process c is generated by a trading strategy (φ, θ) with initial investment w if the following accounting identities are satisfied:

$$\text{(i)}\ \ \theta_t\cdot(S_t + \Delta D_t) + \varphi_t\cdot B_t = w + \int_0^t \theta_u\cdot \mathrm{d}G_u + \int_0^t \varphi_u \mathrm{d}B_u - c_{t-},\ \ 0 \le t \le T,$$

$$\text{(ii)}\ \ \theta_T\cdot(S_T + \Delta D_T) + \varphi_T B_T = \Delta c_T \qquad (3.1.2)$$

We denote by $(\varphi, \theta) \in \Lambda(w, c, S, B, D)$ the statement that (φ, θ) finances c under the financial market (S, B, D) with initial investment w. For technical reasons, we will only consider cumulative cash flow processes c that are square-integrable under the given probability P, i.e.

$E(|c_t|^2) < +\infty$ and $E(|c_{t-}|^2) < +\infty$, $\forall\, t \in [0, T]$

We denote by $\mathcal{C}^2(P)$ the space of all *CCF* processes that are square-integrable under P. Under the above setup, and subject to some regularity conditions, we are able to establish two important facts, namely, existence of an equivalent martingale measure Q for the discounted stock price process (via *Girsanov* transformation) and market completeness.

Lemma 3.1.3 Let $c \in \mathcal{C}^2(P)$ and let $w \in \mathbb{R}$. Then $\exists\ (\alpha, \theta) \in \Lambda(w, c, S, B, D)$ iff there is some φ' s.t. $(\theta, \varphi') \in \Lambda(w, c', S, B, D)$, where $c'(t) = 0$ for all $t < T$ and

$$c_T' = B_T \int_0^T \frac{1}{B_u} \mathrm{d}c_u$$

In other words,

$$\theta_t \cdot S_t + \varphi_t' \cdot B_t = w + \int_0^t \theta_u \cdot \mathrm{d}G_u + \int_0^t \varphi_u' \mathrm{d}B_u,\ 0 \le t \le T,$$

$$\theta_T \cdot S_T + \varphi_T' \cdot B_T = B_T \int_0^T \frac{1}{B_u} \mathrm{d}c_u$$

proof See Appendix B.

QED

Rem The new strategy (θ, φ') can be interpreted as follows. It does noting to the original investment in risky stocks. But instead of consuming c_t at time t, the new investment plan φ' saves the consumption and put it in the bank account (the bond) until T at which the saved consumption grows (due to continuous compounding at the interest rate) to the level $B_T \int_0^T \frac{1}{B_u} \mathrm{d}c_u$ and is consumed finally.

As suggested in the above remark, a natural discount factor in this financial economy is the rate at which the bond grows. So let

$$\hat{G} = S/B + \int \frac{1}{B} \mathrm{d}D$$

be the discounted gain process of the stocks. (The discounted gain process of the bond is of course 1).

Lemma 3.1.4 Let $\hat{G}$ be the discounted gain process of the stocks. Let $c \in \mathcal{L}^2(P)$ and $\theta \in \mathcal{H}^2(G; P) = \mathcal{H}^2(\hat{G}; P)$. Then $\exists\ \varphi \in \mathcal{L}(B; P)$ s.t. $(\theta, \varphi) \in \Lambda(w, c, S, B, D)$ iff $\exists\ \varphi'$ s.t.

$$\theta_t \cdot S_t/B_t + \varphi_t' = w/B_0 + \int_0^t \theta_u \cdot \mathrm{d}\hat{G}_u, \quad 0 \le t \le T,$$

$$\theta_T \cdot S_T/B_T + \varphi_T' = \int_0^T \frac{1}{B_u} \mathrm{d}c_u$$

proof Follows directly from **Lemma 3.1.3** and numeraire invariance.

QED

Notice that $\hat{G}$ is again of the form

$$d\hat{G} = \hat{a}\cdot\psi\cdot dA + \hat{b}\cdot dM,$$

where $\hat{a} = (a_S + a_D)/B + S\cdot a_B$, $\hat{b} = (b_S + b_D)/B$.

A3.1.4 For every $\mathbb{R}^K$-valued predictable process d, the equation $\hat{b}\cdot x = d$ has an $\mathbb{R}^K$-valued predictable solution x. In particular, the equation $\hat{b}\cdot\eta = \hat{a}\cdot\psi$ has an $\mathbb{R}^K$-valued predictable solution η in $\mathcal{L}^2_K(M)$ s.t. the stochastic exponential Z of $X = \{-\int_0^t \eta_u\cdot dM_u\}_{t\in[0,T]}$ is a square-integrable $\mathbb{F}$-martingale under P.

Rem The above requirement of η is satisfied if, for example, $\hat{b}\cdot\eta = \hat{a}\cdot\psi$ has a solution $\eta \in \mathcal{L}^2_K(M)$ s.t.

$$E\left[\int_0^T Z^2 d\langle X\rangle\right] < +\infty$$

since in this case $Z = 1 + \int Z dX$ must be a square-integrable martingale. A more delicate condition for Z to be a square-integrable martingale is the following:

$$\sup_\tau E\left[\exp\left(-\frac{1}{2-\sqrt{2}}\int_0^\tau \theta_u\cdot dM_u\right)\right] < +\infty$$

where the supremum is taken over all stopping times. (See *Norihiko Kamaki* **Theorem 1.5** on Pg 8.)

To show the existence of an equivalent martingale measure of $\hat{G}$ as well as the dynamical completeness of the asset markets, we need the concept of *Girsanov* transformation. In our present framework of the financial economy with market generator M, let X be the process $\{-\int_0^t \eta_u \cdot dM_u\}_{t\in[0,T]}$, which is of course a local martingale under P. Consider the stochastic exponential Z of X. We have the following fact.

Fact 3.1.5 Suppose $\eta = (\eta_1, ..., \eta_K)^\top \in \mathcal{L}_K^2(M)$ is such that the stochastic exponential Z of $\{-\int_0^t \eta_u \cdot dM_u\}_{t\in[0,T]}$, which is given by.

$$Z_t = \exp[\int_0^t \eta_u \cdot dM_u - \frac{1}{2}\Sigma_{k=1}^K \int_0^t \eta_u^{(k)} d\langle M^{(k)}\rangle_u], \forall t \in \mathbb{R}_+$$

is a martingale under P. Then $\frac{dQ}{dP} = Z_T$ defines an equivalent probability measure Q. Let $\hat{M}^{(k)} = M^{(k)} + \int \eta^{(k)} d\langle M^{(k)}\rangle$. Then $\hat{M} = (\hat{M}^{(1)}, ..., \hat{M}^{(k)})^\top$ is a local F-martingale under Q. Moreover, $\hat{M}$ has the representation property under Q.

proof See Appendix B.

QED

Rem the proof of the representation property of $\hat{M}$ gives the *Diffusion Invariance Principle*: if N is a local martingale under P with diffusion coefficient θ w.r.t M, then $\hat{N} := \Gamma(N)$ has the same diffusion coefficient θ w.r.t $\hat{M}$.

Now let Θ be the space consisting of all trading strategies (θ, φ) s.t. $\int \theta \, \mathrm{d}\hat{G}$ is exactly a martingale under Q. We are in a position to show that the market (S, B, D) is complete w.r.t $(\Theta, \mathcal{C}^2(P))$, i.e. for every $c \in \mathcal{C}^2(P)$, there is $(\theta, \varphi) \in \Theta$ replicating c under (S, B, D), subject to some initial investment w.

Prop 3.1.6 Assume **A3.1.1** through **A3.1.4**. with η the required solution to the equation $b \cdot \eta = a \cdot \psi$. Then there is an equivalent probability measure Q under which

$$\hat{M}^{(k)} := M^{(k)} + \int \eta^{(k)} \mathrm{d}\langle M^{(k)} \rangle$$

is a local F-martingale and $\hat{M} = (\hat{M}^{(1)}, \ldots, \hat{M}^{(K)})^{\top}$ has the representation property. Also

$$\mathrm{d}\hat{G}_t = b_t \cdot \mathrm{d}\hat{M}_t$$

and $\hat{G}$ is actually a martingale under Q. Moreover, the asset market is dynamically complete w.r.t $(\Theta, \mathcal{C}^2(P))$, i.e. for every future *CCF* process $c \in \mathcal{C}^2(P)$, there is some initial investment level w and some trading strategy $(\theta, \varphi) \in \Theta$ that replicates c with initial investment w.

proof See Appendix B.

QED

In order to define arbitrage opportunities, we need to endow $\mathcal{L}^2(P)$ with some ordering. For two processes x, y in $\mathcal{L}^2(P)$, we denote by $x \succsim y$ the following conditions

$$x_t \geq y_t,\ P\text{-a.s.},\ \forall\, t \in [0, T]$$

Also we denote by $x \succ y$ the following condition

$$x \succsim y \text{ and there is some } t \in [0, T] \text{ s.t. } P\{x_t > y_t\} > 0$$

Following *Steven Ross*, we call a trading strategy (φ, θ) an arbitrage strategy if it generates some *CCF* c with initial investment w s.t. either (i) $w \leq 0$ but $c \succ 0$, or (ii) $w < 0$, but $c \succsim 0$. We may combine (i) and (ii) and get $c + w \succ 0$

Cor 3.1.7 The space Θ contains no arbitrage under the market (S, B, D). Moreover, for every $c \in \mathcal{L}^2(P)$, and for every $(\theta, \varphi) \in \Theta$ replicating c (with some initial investment), we have

$$\theta_t \cdot S_t/B_t + \varphi_t = E^Q[\Delta c_t/B_t + \int_t^T \frac{1}{B_u} \mathrm{d}c_u \mid \mathcal{F}_t],\ \forall\, t \in [0, T] \quad (3.1.3)$$

proof See Apendix B

QED

Next, suppose we add a new security to the market. Suppose this new security is a claim to the contingent payoff represented by some future *CCF* process $D^{(N+1)} \in \mathcal{L}^2(P)$. The valuation of $D^{(N+1)}$ is given in the following.

Cor 3.1.8 Let $D^{(N+1)} \in \mathcal{L}_0^2(P)$ be some contingent claim to some square-integrable future *CCF* process. and let $S^{(N+1)}$ be its corresponding ex-dividend price process, an *RCLL* semi-martingale. Let Π be a space of trading strategies for the expanded market s.t.

(i) $\forall\ (\theta, \theta^{(N+1)}, \varphi) \in \Pi,\ I_{A\times(s,\ T]}\cdot(\theta, \theta^{(N+1)}, \varphi) \in \Pi$

(ii) Π contains Θ and contains the strategy $(\theta, \theta^{(N+1)}, \varphi)$ with $\theta = 0,\ \ \varphi = 0$ and $\theta^{(N+1)} = 1$.

If Π continues to be free of arbitrage under the expanded market $(S, B, S^{(N+1)}, D, D^{(N+1)})$, then

$$S_t^{(N+1)} = B_t \cdot E^Q[\int_t^T \frac{1}{B_u} \mathrm{d}D_u^{(N+1)} \mid \mathcal{F}_t] \qquad (3.1.4)$$

proof By **Prop 3.1.6**, there is some $(\theta, \varphi) \in \Theta$ financing $D^{(N+1)}$ with some initial investment level w. On the other hand, the trading strategy that only has constant 1 as the position on the new security (and zero on all other securities) also finances $D^{(N+1)}$. Thus both will have the same market values at all time. By the numeraire invariance,

$$S_t^{(N+1)}/B_t + \Delta\hat{D}_t^{(N+1)} = \theta_t \cdot S_t/B_t + \varphi_t = \hat{D}_T^{(N+1)} - \hat{D}_{t-}^{(N+1)} - \int_t^T \theta_u \mathrm{d}\hat{G}_u$$

Taking expectation under Q conditional on $\mathcal{F}_t$ yields

$$S_t^{(N+1)}/B_t + \Delta\hat{D}_t^{(N+1)} = E^Q(\hat{D}_T^{(N+1)} - \hat{D}_{t-}^{(N+1)} \mid \mathcal{F}_t), \text{ or}$$

$$S_t^{(N+1)} = B_t \cdot E^Q(\hat{D}_T^{(N+1)} - \hat{D}_t^{(N+1)} \mid \mathcal{F}_t) = B_t \cdot E^Q[\int_t^T \frac{1}{B_u} dD_u^{(N+1)} \mid \mathcal{F}_t].$$

QED

3.2 Arbitrage Pricing of Derivative Securities

For every $t \in [0, T]$ fixed, let Z be square-integrable and $\mathcal{F}_t$-measurable random variable and suppose $c \in \mathcal{L}^2(P)$ is given by $c_s = 0$ for $s < t$ and $c_s = Z$ for $s \geq t$. In other words, c delivers nothing except the lump-sum Z at t (as in the case of a zero-coupon bond that matures at t). Let $(\theta, \varphi) \in \Theta$ replicate c. Then from (4.1.3) we have

$$\theta_s \cdot S_s/B_s + \varphi_s = E^Q[Z/B_t \mid \mathcal{F}_s],\ 0 \leq s \leq t \ \ (\text{and } \theta_s \cdot S_s/B_s + \varphi_s = 0 \text{ for } s > t)$$

or,

$$\theta_s \cdot S_s + \varphi_s = E^Q\left(\frac{B_s}{B_t} \cdot Z \middle| \mathcal{F}_s \right) \quad 0 \leq s \leq t \qquad (3.2.1)$$

Suppose B is deterministic (a riskless bond). Then we get

$$\frac{B_t}{B_s} = E^Q\left(\frac{Z}{\theta_s \cdot S_s + \varphi_s} \middle| \mathcal{F}_s \right) \quad 0 \leq s \leq t \qquad (3.2.2)$$

The L.H.S of (3.2.2) is the return on bond over the holding period $[s, t]$. The R.H.S is the conditional expected return on the portfolio (θ, φ) over the holding period $[s, t]$ under the risk-neutral probability Q.

A particular case to keep in mind is when B_t is given by

$$B_t = B_0 \cdot e^{\int_0^t r_u \psi_u \mathrm{d}A_u},$$

where r is a continuous (and often strictly positive) $\mathbb{R}^{1\times K}$-valued semimartingale. In this case r is referred to as the short-term interest rate or simply the short rate representing the rate at which the bond value is continuously compounding. In this case, (3.2.1) becomes

$$\theta_s \cdot S_s + \varphi_s = E^Q\left(e^{\int_s^t -r_u \psi_u \mathrm{d}A_u} \cdot Z \,\middle|\, \mathcal{F}_s \right), \qquad 0 \le s \le t \qquad (3.2.3)$$

The above provides risk-neutral valuations of derivative securities, which will be discussed below.

We first look at forward prices. A forward contract issued at time t with maturity T is a contract that specifies a contingent payment W_T to the holder (the long party) in exchange for a known fee F_t upon maturity. The contingent payment W_T is unknown until after information $\mathcal{F}_T$ is realized so W_T is $\mathcal{F}_T$-measurable square-integrable random variable. On the other hand, F_t is specified at t (although it is paid to the short party at time T) so it is F_t-measurable. By convention, F_t is set to such a value that the initial value of the contract is exactly 0 at the time of issuance (i.e. at t). In this case, F_t is called the forward price of the contract. Now to the long party, the forward contract represents a CCF c given by

$$c_s = \begin{cases} 0, & \text{if } s < T; \\ W_T - F_t, & \text{if } s = T \end{cases}$$

By the definition of F_t and (3.2.3), it follows that we have

$$0 = E^Q\left(e^{\int_t^T -r_u \psi_u \mathrm{d}A_u} \cdot (W_T - F_t) \middle| \mathcal{F}_t \right) \tag{3.2.4}$$

It follows from (2.4) that we have

$$F_t = \frac{1}{P(t,T)} \cdot E^Q\left(e^{\int_t^T -r_u \psi_u \mathrm{d}A_u} \cdot W_T \middle| \mathcal{F}_t \right) \tag{3.2.5}$$

where $P(t, T)$ is the time-t price of the discount bond that pays one unit at time T. For example, suppose the forward contract is made on an underlying portfolio θ of equity securities with market value $S_T \bullet \theta_T$ at time T so that $W_T = S_T \bullet \theta_T$. Also suppose θ generates a cumulative cash flow stream $c(\theta)$. Then from (3.1.3) we have

$$\theta_t \cdot S_t = B_t E^Q[\Delta c_t / B_t + \int_t^T \frac{1}{B_u} \mathrm{d}c_u(\theta) \mid \mathcal{F}_t] = B_t E^Q[\Delta c_t / B_t + \int_t^T \frac{1}{B_u} \mathrm{d}c_u(\theta) \mid \mathcal{F}_t] =$$

$$E^Q\left(e^{\int_t^T -r_u \psi_u \mathrm{d}A_u} \cdot W_T \middle| \mathcal{F}_t \right) + E^Q\left(\Delta c_t / B_t + \int_t^{T-} \frac{1}{B_s} \mathrm{d}c_s \middle| \mathcal{F}_t \right) \Rightarrow$$

$$F_t = P(t, T)^{-1} \cdot S_t \cdot \theta_t - P(t, T)^{-1} \cdot E^Q\left(\Delta c_t / B_t + \int_t^{T-} \frac{1}{B_s} \mathrm{d}c_s \middle| \mathcal{F}_t \right) \quad (3.2.6)$$

Let $DI_{[t,\,T)} = \hat{c}_{T-} - \hat{c}_{t-} = \Delta c_t / B_t + \int_t^{T-} \frac{1}{B_s} \mathrm{d}c_s(\theta)$ be the discounted cumulative cash inflow of the underling portfolio from t (including jump at t) up to T (but not including the final jump at T). Then (3.2.6) is rewritten as

$$F = P(t, T)^{-1} [S_t \cdot \theta_t - E^Q(DI_{[t,\,T)} \,|\, \mathcal{F}_t)] \quad (3.2.7)$$

(3.2.7) is called the *cost-of-carry* formula for forward prices. A special case of (3.2.7) is worth noting. Suppose the cumulative cash flow c and the deflator process are deterministic given $\mathcal{F}_t$. Then the computation is independent of the equivalent martingale measure Q and we have

$$F_t = e^{\int_t^T -r_u \psi_u \mathrm{d}A_u} \cdot \{S_t \cdot \theta_t - DI_{[t,\,T)}\} \quad (3.2.8)$$

Next, we consider arbitrage pricing of American type options. More generally, we consider valuation of American securities. An American security is described by an adapted process $Y = \{Y_t\}_{t\in[0,\ T]}$ together with a subset $F \subseteq [0, T]$. Y_t will be interpreted as the contingent payoff to the holder if he (she) chooses to exercise his (her) claim at time t. Also F represents the time windows available to the holder of the security to exercise his (her) right. For example, a typical American call option on a security with strike price K and maturity T is given by $Y_t = (S_t - K)^+$, $F = [0, T]$. A *European* option is also a special case in that $F = \{T\}$. In this context, an exercise policy is represented by an F-stopping time τ: $\Omega \to F$ so that $\tau(\omega)$ is the exercise time when ω is the outcome. For a fixed exercise policy τ, the contingent payoff c of the American security associated with this policy is given by

$$c_t = \begin{cases} Y_t, & \text{if } t = \tau; \\ 0, & \text{otherwise} \end{cases}$$

It follows from the risk-neutral valuation formula, the arbitrage free value of the American security associated with τ is given by

$$V_t(\tau) = E^Q\,[e^{\int_t^\tau -r_u \psi_u \mathrm{d}A_u} \cdot Y_\tau \,|\, \mathcal{F}_t]$$

Let $\Lambda(t)$ be the set of all F-stopping times $\tau : \Omega \to F$ with $\tau \geq t$. At time t, a rational agent will of course choose an exercise policy $\tau \in \Lambda(t)$ to maximize $V_t(\tau)$ and this suggest that the "fair" value of the American security is given by

$$V_t = \sup_{\tau\in\Lambda(t)} E^Q\,[e^{\int_t^\tau -r_u \psi_u \mathrm{d}A_u} \cdot Y_\tau \,|\, \mathcal{F}_t] \tag{3.2.9}$$

We show that (3.2.9) does indeed give a arbitrage-free value of the American security. Let V_t be the actual market price at t of the American security. First, suppose

$$V_t < \sup_{\tau\in\Lambda(t)} E^Q\,[e^{\int_t^\tau -r_u\psi_u \mathrm{d}A_u} \cdot Y_\tau | \mathcal{F}_t].$$

Then there exists an exercise policy $\tau \in \Lambda(t)$ s.t.

$$V_t < E^Q\,[e^{\int_t^\tau -r_u\psi_u \mathrm{d}A_u} \cdot Y_\tau | \mathcal{F}_t].$$

Now an investor can purchase the American security for V_t and choose exercise policy τ. On the other hand, using stocks and bond in the primitive asset markets, he (or she) can replicate the short position of the American security associated with the given policy τ for that has an initial price $E^Q\,[e^{\int_t^\tau -r_u\psi_u \mathrm{d}A_u} \cdot Y_\tau | \mathcal{F}_t]$. So he (she) gets positive cash

$$E^Q\,[e^{\int_t^\tau -r_u\psi_u \mathrm{d}A_u} \cdot Y_\tau | \mathcal{F}_t] - V_t > 0$$

His (her) future obligation is exactly zero by the two mutually offsetting portfolios. This is obviously an arbitrage. Similarly, if $V_t > E^Q\,[e^{\int_t^\tau -r_u\psi_u \mathrm{d}A_u} \cdot Y_\tau | \mathcal{F}_t]$, the one can construct an arbitrage. This argument shows that the arbitrage free value of the American security should be as given in (3.2.9).

Although we have just dealt with a few examples of derivative securities, it should be clear that the application of the risk-neutral valuation is by no means limited to what we have discussed above.

4 Equity Option Pricing with Gaussian Term Structures of Interest Rates

4.1 Introduction

The *Black-Scholes* option pricing formula has been hailed as one of the crowning achievements in the theory of finance. Over the years, a number of variants of the pricing formula have been developed to better accommodate data observed in real financial markets. One of the features of the original formula that people find too restrictive is that the short term rate of the bond market is assumed to be fixed. This is especially inconvenient for pricing "interest rate sensitive" products such as bond options, interest rate swaps, swaptions, and interest rate caps and floors (which can be decomposed into a series of bond options), etc.. For this reason, a number of authors have recast the *Black-Scholes* formula in terms of bond options with random movements of short term rates.

Although interest rate risks have received much consideration in the context of pricing of interest rate sensitive products mentioned above, this should by no means imply that equity options are not (or less) sensitive to interest rate movements. In fact, for long-lived equity options, total ignorance of the interest rate risks may result in serious mispricing.[4] This is not only because that the longer the investment horizon, the more chances there are for the interest rate to fluctuate, but also because the length of the holding period may serve to exaggerate any interest rate errors as is suggested in the original *Black-Scholes* option pricing formula for a call option on a stock with current price S, strike K and expiration T:

[4] Some trading experts on the Wall Street suggest that the derivatives disasters of some heavily leveraged firms are largely due to sudden movements in short-term interest rates.

$$V(t) = S\Phi(d_1) - e^{-r(T-t)}.K\Phi(d_2)$$

where $\Phi(\cdot)$ denotes the c.d.f. of the standard normal distribution and

$$d_1 = \frac{\ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\ln(S/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma$$

$\sqrt{T-t}$

In the expressions for d_1 and d_2 we see that any errors in r are amplified by a factor of $\sqrt{T-t}$.

In this paper, we develop a sophisticated equity option pricing formula by incorporating term structures of interest rates. The term structure we consider here assumes that the short rate r is a stochastic process generated by the following SDE:

$$dr(s) = [a(s)r(s) + b(s)]ds + c(s)d\hat{W}(s), \text{ subject to } r(0) \text{ given} \qquad (4.1.1)$$

where a, b , c are continuous deterministic functions of the time s and $\hat{W}$ is the standard *Brownian* motion under some risk-neutral probability measure Q. The term structure is called *Gaussian* because the short rate process is normally distributed. We do not specify functional forms of a, b, c so that it is broad enough to include the popular models such as the *Vasicek* term structure or the "arbitrage-free" models such as *Hull-White* and *Ho-Lee*. Also, by allowing free forms of a, b and c we may calibrate these parameters so that the term structure (i.e. the set of prices of default-free discount bonds with varying maturities into the future) best fit the yield curves of the bond markets. For an equilibrium justification of the *Gaussian* term structure models, the reader may consult, for example, *Darrell Duffie*, *Costis Skiadas et al* (1995) in the framework of pure security exchange economy with a representative agent with recursive utility.

4.2 The Model

We begin by assuming a stochastic base $(\Omega, \mathcal{J}, P, \mathsf{F})$ where $(\Omega, \mathcal{J}, P)$ is a complete probability pace and $\mathsf{F} = \{\mathcal{J}_t\}_{t\geq 0}$ is a filtration on the probability space satisfying the usual conditions. Upon the stochastic base, the primitive asset markets consist of a non dividend-paying stock whose price process S is given by

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t)$$

and a money market investment opportunity characterized by the value process

$$dB(t) = B(t)r(t)dt, \text{ subject to } B(0) \text{ given}$$

where $\mu(t)$, $\sigma(t)$ are continuous deterministic functions of t, and $r(t)$ is a continuous stochastic process to be specified later on, and W is a one-dimensional F-*Brownian* motion. Under some mild restrictions (e.g. assume that μ, σ are bounded), we obtain, using the *Girsanov* transformation argument, an equivalent martingale measure Q and a new standard F-*Brownian* motion $\hat{W}$ under Q s.t.

$$dS(s) = r(t)S(t)dt + \sigma(t)S(t)d\hat{W}(t)$$

We assume that under this risk-neutral valuation, the short rate process r is given by (1.1). Notice also that the discounted price process

$$S^*(t) \equiv S(t) \cdot e^{-\int_0^t r(\tau)\,d\tau}$$

is a martingale under Q (hence the name martingale measure for Q) and we thus have

$$S(t) = E^Q[S(T)e^{-\int_t^T r(\tau)\,d\tau} \mid \mathcal{F}_t] \qquad (4.2.2)$$

More generally, The risk-neutral valuation implies that the market is dynamically complete and every contingent claim can be priced. Specifically, the arbitrage-free value $V(t)$ at time t of the claim to a random payoff Z, an $\mathcal{F}_t$-measurable and Q-integrable random variable delivered at time $T \geq t$ is given by

$$V(t) = E^Q[e^{-\int_t^T r(\tau)\,d\tau} \cdot Z \mid \mathcal{F}_t]$$

In particular, the arbitrage-free value $P(t, T)$ at time t of a default-free discount bond paying one unit at time T is given by

$$P(t, T) = E^Q[e^{-\int_t^T r(\tau)\,d\tau} \mid \mathcal{F}_t] \qquad (4.2.3)$$

Similarly, the arbitrage-free value $V(t)$ at time t of a call option on the equity with maturity T and strike price K is given by

$$V(t) = E^Q\left((S(T) - K)^+ \cdot \exp\left(-\int_t^T r(s)\mathrm{d}s \right) \middle| \mathcal{J}_t \right) \qquad (4.2.4)$$

The complications with trying to develop an analytical pricing formula for (4.2.4) lie in that we can no longer take the discount factor directly outside of the expectation operator since it is stochastic and not $\mathcal{J}_t$-measurable. Thus we need to know the joint distribution of both $S(T)$ and the above discount factor. Let

$$y(s) = \int_0^s r(\tau)\mathrm{d}\tau, \ \forall\, s \geq 0.$$

Then

$$y(t) - y(T) = -\int_t^T r(s)\mathrm{d}s$$

$$V(t) = E^Q[(S(T) - K)^+ . e^{y(t) - y(T)} \mid \mathcal{J}_t]$$

Also, let $z(s) = \ln S(s)$. Then we have, by *Ito*'s lemma,

$$\mathrm{d}z(s) = [r(s) - \frac{1}{2}\sigma^2(s)]\mathrm{d}s \ + \sigma(s)\mathrm{d}\hat{W}(s)$$

and we get

$$V(t) \ = E^Q[(e^{z(T)} - K)^+ . e^{y(T)} \mid \mathcal{J}_t] = E^Q[1_{\{z(T) \geq \ln K\}} \cdot (e^{z(T) + y(t) - y(T)} - e^{y(t) - y(T)} . K)^+ . \mid \mathcal{J}_t]$$

The key to evaluating the above expression is recognizing that the vector $(z(T), y(T))$ is bivariate normal conditional on $\mathcal{F}_t$. The analysis is based on the following fact about linear SDE.

Fact 4.2.1 Let $X = \{X(s)\}_{s\in[0,\infty)}$ be an N-dimensional random process generated by the following SDE

$$dX(s) = [A(s)X(s) + B(s)]ds + C(s)dW(s), \text{ s.t. } X(0) \text{ given}$$

where $A(s)$ is an $N\times N$ matrix-valued continuous function of s and $B(s)$ and $C(s)$ are N-dimensional vector-valued continuous functions of s. Then X is a *Gaussian* process, i.e. $X(s_1), X(s_2), ..., X(s_n)$ are jointly normal for every time sequence $s_1 \le s_2 \le \cdots \le s_n$ given. Moreover, X is *Markovian* w.r.t F and for every $t \le T$, given, the distribution of $X(s)$ conditional on $X(t)$ (and hence conditional on $\mathcal{F}_t$ due to the *Markov* property) is multivariate normal with the mean vector m and the variance-covariance matrix Σ satisfying the following initial value problems on the time interval $[t, \infty)$:

$$dm(s) = [A(s)m(s) + B(s)]ds, \text{ s.t. } m(t) = X(t)$$

$$d\Sigma(s) = [A(s)\Sigma(s) + \Sigma(s)A^T(s) + C(s)C^T(s)]ds, \text{ s.t. } \Sigma(t) = 0$$

For reference to the above result, the reader may consult with any standard text on SDE such as *Karatzas & Shreve* (1988).

Now, if we let

$$A=\begin{pmatrix} a & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} b \\ 0 \\ -\frac{1}{2}\sigma^2 \end{pmatrix}, \quad C=\begin{pmatrix} c \\ 0 \\ \sigma \end{pmatrix}$$

and let

$$X=\begin{pmatrix} r \\ y \\ z \end{pmatrix}$$

Then, X satisfies the following linear SDE

$$dX = (AX + B)ds + Cd\hat{W}$$

and thus **Fact 1.1** implies that X is a *Markovian Gaussian* process. Notice that the *Markov* property allows us to replace the information $\mathcal{F}_t$ with $X(t)$ when taking conditional expectations. Thus $V(t)$ depends on current value $X(t)$ (and the expiration time T). Therefore we have

$$V(t) = V(X, t, T) = E^Q[1_{\{z(T) \geq \ln K\}} \cdot (e^{z(T) + y(t) - y(T)} - e^{y(t) - y(T)} . K)^+ . \mid X(t) = X]$$

Lem 4.2.2 Conditional on $X(t) = X$, let ρ be the correlation coefficient of $y(t) - y(T)$ and $z(T)$ and let

$$z(T) \sim N(a_1, b_1^2) \text{ and } y(t) - y(T) \sim N(a_2, b_2^2)$$

let $b_{12} \equiv \rho b_1 \cdot b_2 = Cov(z(T), y(T) \mid X(t))$. Let $P(t, T)$ be the price at time t of the default-free discount bond paying one unit at time T. Let $V(t)$ be the price of a *European* call option on the underlying stock with strike K and exercise date T. Then

(4.2.5) $$P(t, T) = \exp\left(\frac{1}{2}b_2^2 + a_2\right)$$

and

(4.2.6) $$V(t) = S(t)\Phi(d_1) - P(t, T)\cdot K\cdot\Phi(d_2)$$

where

$$d_1 = -\frac{\ln K - a_1 - b_{12} - b_1^2}{b_1}, \quad d_2 = -\frac{\ln K - a_1 - b_{12}}{b_1} = d_1 - b_1$$

proof See Appendix C.

QED

Notice the similarity between (4.2.6) and the classical *Black-Scholes* formula. In the latter, owe also have the relation $d_2 = d_1 - \sigma\sqrt{T-t}$, where $\sigma\sqrt{T-t}$ is exactly b_1 in our model when σ and r are both constant. Also, if the short rate r is fixed (i.e. $a = b = c = 0$), then the discount bond price is exactly

$$P(t, T) = e^{-r(T-t)}$$

This is also very similar to the bond option pricing formula developed by *Jamshidian* using the *Vasicek* term structure. We can see that equity option valuation is indeed very much dependent upon interest rate movements because the pricing formula is closely tied to bond prices. The difference here is that the underlying equity prices provide a new dimension of random movements and thus the option price turns out to depend on two explanatory factors, namely, current short-term rate as well as current stock price.

Next we deal with computation of the parameters a_i, b_i and b_{12}. Of course we may directly apply **Fact 4.2.1** to the 3-dimensional SDE to find out the joint mean and the variance-covariance. But the evaluation is very cumbersome. On the other hand we find it much easier to work with 2-dimensional systems.

First, observe that the vector (r, z) arises as the solution to

$$\mathrm{d}\begin{pmatrix} r \\ z \end{pmatrix} = [A_1\begin{pmatrix} r \\ z \end{pmatrix} + B_1]\mathrm{d}s + C_1\mathrm{d}\hat{W}$$

where

$$A_1 = \begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} b \\ -\frac{1}{2}\sigma^2 \end{pmatrix}, \quad C_1 = \begin{pmatrix} c \\ \sigma \end{pmatrix}$$

Hence the mean vector (μ_1, μ_2) and the variance-covariance matrix Σ_1 are given by

(4.2.7) $$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = A_1 \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + B_1 \text{ subject to } \mu_1(t) = r(t), \text{ and } \mu_2(t) = z(t)$$

$$\mathrm{d}\Sigma_1/\mathrm{d}s = A_1\Sigma_1 + \Sigma_1 A_1^{\mathrm{T}} + C_1 C_1^{\mathrm{T}} \text{ subject to } \Sigma_1(t) = 0$$

Solving the above yields formulas for a_1 and b_1.

Also, notice that the vector (r, y) jointly satisfies the following SDE

$$\mathrm{d}\begin{pmatrix} r \\ y \end{pmatrix} = [A_2\begin{pmatrix} r \\ y \end{pmatrix} + B_2]\mathrm{d}s + C_2\mathrm{d}\hat{W}$$

where

$$A_2 = \begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} c \\ 0 \end{pmatrix}$$

The variance-covariance matrix Σ_2 satisfies the following initial value problem on the time interval $[t, \infty)$:

(4.2.8) $$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = A_2 \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + B_2 \text{ subject to } m_1(t) = r(t), \text{ and } m_2(t) = y(t)$$

$$d\Sigma_2/ds = A_2\Sigma_2 + \Sigma_2 A_2^{\mathrm{T}} + C_2 C_2^{\mathrm{T}} \text{ subject to } \Sigma_2(t) = 0$$

Solving (4.2.8) yields formulas for a_2 and b_2.

Next, let $x = z + y$ and consider the vector (r, x) which arises as the solution to

$$d\begin{pmatrix} r \\ x \end{pmatrix} = [A_3\begin{pmatrix} r \\ x \end{pmatrix} + B_3]ds + C_3 d\hat{W}$$

where

$$A_3 = \begin{pmatrix} a & 0 \\ 2 & 0 \end{pmatrix}, \quad B_3 = B_2 = \begin{pmatrix} b \\ -\frac{1}{2}\sigma^2 \end{pmatrix}, \quad C_3 = C_2 = \begin{pmatrix} c \\ \sigma \end{pmatrix}$$

Hence the variance-covariance matrix Σ_3 of (r, x) are solutions to the following linear system:

(4.2.9) $d\Sigma_3/ds = A_3\Sigma_3 + \Sigma_3 A_3^{\mathrm{T}} + C_3 C_3^{\mathrm{T}}$ subject to $\Sigma_3(t) = 0$

Solving the above yields, among other things, the formula for $b_3^2 \equiv Var(x)$.

Finally, using the relation $Var(x) = Var(z) + 2Cov(z, y) + Var(y)$, we obtain the formula for the covariance parameter b_{12}:

$$b_{12} = [Var(x) - Var(z) - Var(y)]/2 = [b_3^2 - b_2^2 - b_1^2]/2$$

and we thus have the following lemma:

Lemma 4.2.3 With the above notation, we have,

$$a_1 = z(t) + r(t)\int_t^T \varphi(s)\mathrm{d}s + \int_t^T \varphi(s)\left[\int_t^s \frac{b(\bar{t})}{\varphi(\bar{t})}\mathrm{d}\bar{t}\right]\mathrm{d}s - \frac{1}{2}\int_t^T \sigma^2(s)\mathrm{d}s$$

$$a_2 = -r(t)\int_t^T \varphi(s)\mathrm{d}s - \int_t^T \varphi(s)\left[\int_t^s \frac{b(\bar{t})}{\varphi(\bar{t})}\mathrm{d}\bar{t}\right]\mathrm{d}s$$

$$b_1^2 = 2\int_t^T \varphi(s)[f(s)+g(s)]\mathrm{d}s + \int_t^T \sigma^2(s)\mathrm{d}s$$

$$b_2^2 = 2\int_t^T \varphi(s)f(s)\mathrm{d}s$$

$$b_3^2 = 4\int_t^T \varphi(s)[2f(s)+g(s)]\mathrm{d}s + \int_t^T \sigma^2(s)\mathrm{d}s$$

$$b_{12} = [b_3^2 - b_1^2 - b_2^2]/2 = \int_t^T \varphi(s)[2f(s)+g(s)]\mathrm{d}s$$

where

$$\varphi(s) = \exp\{\int_t^s a(\bar{s})\mathrm{d}\bar{s}\}$$

$$f(s) = \int_t^s \left[\varphi(\bar{s}) \cdot \int_t^{\bar{s}} \left[\frac{c(\bar{t})}{\varphi(\bar{t})}\right]^2 \mathrm{d}\bar{t}\right]\mathrm{d}\bar{s}$$

$$g(s) = \int_t^s \frac{\sigma(\bar{s})c(\bar{s})}{\varphi(\bar{s})} \mathrm{d}\bar{s}$$

proof For details see Appendix C.

QED

Now we summarize the lemmas into the following proposition.

Prop 4.2.4 With the above setup, the price $P(t, T, \bar{r})$ at time t of the default-free discount bond paying one unit at time T, given $r(t) = \bar{r}$, is given by

$$P(t, T, \bar{r}) = \exp[H(t, T, \bar{r})]$$

and $H(t, T, \bar{r})$ is given by

$$H(t, T, \bar{r}) = \int_t^T \varphi(s)[f(s) - h(s) - \bar{r}]\mathrm{d}s$$

where

$$\varphi(s) = \exp\int_t^s a(\tau)\mathrm{d}\tau$$

$$h(s) = \int_t^s \frac{b(\tau)}{\varphi(\tau)} \mathrm{d}\tau$$

$$f(s) = \int_t^s \left[\varphi(\bar{s}) \cdot \int_t^{\bar{s}} \left[\frac{c(\bar{t})}{\varphi(\bar{t})} \right]^2 \mathrm{d}\bar{t} \right] \mathrm{d}\bar{s}$$

The price $V(t, T, \bar{S}, \bar{r})$ of a *European* call option at time t which expires at T with strike K, given that the stock price is $\bar{S}$ and the short rate is $\bar{r}$ at t, is given by

$$V(t, T, \bar{S}, \bar{r}) = \bar{S}\,\Phi(d_1) - P(t, T, \bar{r})\Phi(d_2)$$

$$d_1 = \frac{\ln(\bar{S}/K) + \int_t^T \varphi(s)[\bar{r} + h(s) + 4f(s) + 3g(s)]\mathrm{d}s + (1/2)\int_t^T \sigma^2(s)\mathrm{d}s}{\left\{ \int_t^T 2\varphi(s)[f(s) + g(s)]\mathrm{d}s + \int_t^T \sigma^2(s)\mathrm{d}s \right\}^{1/2}}$$

$$d_2 = \frac{\ln(\bar{S}/K) + \int_t^T \varphi(s)[\bar{r} + h(s) + 2f(s) + g(s)]\mathrm{d}s - (1/2)\int_t^T \sigma^2(s)\mathrm{d}s}{\left\{ \int_t^T 2\varphi(s)[f(s) + g(s)]\mathrm{d}s + \int_t^T \sigma^2(s)\mathrm{d}s \right\}^{1/2}}$$

where $g(s) = \int_t^s \frac{\sigma(\bar{s})c(\bar{s})}{\varphi(\bar{s})}\mathrm{d}\bar{s}$

proof Direct consequence of the above two lemmas.

QED

4.3 Special Cases

We now consider some special cases.

CASE 4.3.1 *The Classical Black-Scholes Formula* Suppose $a = b = c = 0$ so that the short term rate is fixed at $\bar{r}$. Also, suppose the volatility parameter σ of the equity price is also constant. Then clearly,

$$P(t, T) = e^{-\bar{r}(T-t)}$$

Also, we have $f = g = h = 0$ and $\varphi(s) = 1$ so we quickly get

$$d_1 = \frac{\ln(\bar{S}/K) + [\bar{r} + (1/2)\sigma^2](T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(\bar{S}/K) + [\bar{r} - (1/2)\sigma^2](T-t)}{\sigma\sqrt{T-t}}$$

CASE 4.3.2 *The Vasicek Term Structure* Suppose the short rate process r is mean-reverting as in

$$\mathrm{d}r(t) = \alpha(\beta - r)\mathrm{d}t + c\mathrm{d}\hat{W}(t)$$

where $\alpha > 0$, $\beta > 0$ and c are all constants with β representing the equilibrium short rate. Also suppose the volatility σ is constant. Letting $a = -\alpha$ and $b = \alpha\beta$ gives

$$\varphi(s) = \exp[-\alpha(s-t)]$$

$$h(s) = \beta[e^{\alpha(s-t)} - 1]$$

$$g(s) = \frac{\sigma c}{\alpha}[e^{\alpha(s-t)} - 1]$$

$$f(s) = \frac{c^2}{2\alpha^2}[e^{\alpha(s-t)} - 2 + e^{-\alpha(s-t)}]$$

$$H(t,T,\bar{r}) = \int_t^T \varphi(s)[f(s) - h(s) - \bar{r}]ds = \left(\frac{c^2}{2\alpha^2} - \beta\right)(T-t) + \left(\frac{c^2}{\alpha^3} + \frac{\bar{r}-\beta}{\alpha}\right)A(t,T) - \frac{c^2}{4\alpha^3}B(t,T)$$ where

$$A(t, T) = e^{-\alpha(T-t)} - 1, \quad B(t, T) = e^{-2\alpha(T-t)} - 1$$

$$P(t, T, \bar{r}) = \exp\left\{\left(\frac{c^2}{2\alpha^2} - \beta\right)(T-t) + \left(\frac{c^2}{\alpha^3} + \frac{\bar{r}-\beta}{\alpha}\right)A(t,T) - \frac{c^2}{4\alpha^3}B(t,T)\right\}$$

as is exactly given in *Vasicek* (1977) and *Jamshidian* (1989).

Also, we have

$$d_1 = \frac{\ln(\bar{S}/K) + \left(\frac{\beta}{\alpha} + \frac{6c^2}{\alpha^3} - \frac{\bar{r}}{\alpha} + \frac{4\sigma c}{\alpha^2}\right)A(t,T) - \frac{3c^2}{2\alpha^3}B(t,T) + \left(\frac{1}{2}\sigma^2 + \frac{3c^2}{\alpha^2} + \beta + \frac{4\sigma c}{\alpha}\right)(T-t)}{\left\{\left(\frac{c^2}{\alpha^2} + \frac{2\sigma c}{\alpha} + \sigma^2\right)(T-t) + \left(\frac{2c^2}{\alpha^3} + \frac{2\sigma c}{\alpha^2}\right)A(t,T) - \frac{c^2}{2\alpha^2}B(t,T)\right\}^{1/2}}$$

$$d_2 = \frac{\ln(\bar{S}/K) + \left(\frac{\beta}{\alpha} + \frac{4c^2}{\alpha^3} - \frac{\bar{r}}{\alpha} + \frac{2\sigma c}{\alpha^2}\right)A(t,T) - \frac{c^2}{\alpha^3}B(t,T) + \left(-\frac{1}{2}\sigma^2 + \frac{2c^2}{\alpha^2} + \beta + \frac{2\sigma c}{\alpha}\right)(T-t)}{\left\{\left(\frac{c^2}{\alpha^2} + \frac{2\sigma c}{\alpha} + \sigma^2\right)(T-t) + \left(\frac{2c^2}{\alpha^3} + \frac{2\sigma c}{\alpha^2}\right)A(t,T) - \frac{c^2}{2\alpha^2}B(t,T)\right\}^{1/2}}$$

Interestingly, if we let $c = 0$ and let $\alpha \to 0$, then we get, in the limit, the classical *Black-Scholes* formula again.

CASE 4.3.3 *The Ho-Lee Model* The short rate process is given by

$$dr(s) = b(s)d(s) + c\,d\hat{W}(s),$$

where c is a constant. Assume that σ is constant. We then have

$$\varphi(s) = 1$$

$$h(s) = \int_t^s b(\tau)\,d\tau$$

$$f(s) = c^2\frac{(s-t)^2}{2}$$

$$g(s) = \sigma c(s - t)$$

Therefore we have

$$P(t, T, \bar{r}) = A(t, T)\exp[-\bar{r}(T-t)]$$

where

$$A(t, T) = \exp\{ c^2(T-t)^3/6 - B(t, T)\}$$

$$B(t, T) = \int_t^T \left[\int_t^s b(\tau)\,\mathrm{d}\tau \right] \mathrm{d}s$$

Also, we have

$$d_1 = \frac{\ln \bar{S}/K + [\bar{r} + (1/2)\sigma^2 + 2\sigma c(T-t) + c^2(T-t)^2](T-t) + B(t,T)}{[(1/3)c^2(T-t)^2 + \sigma c(T-t) + \sigma^2]^{1/2}\sqrt{T-t}}$$

$$d_2 = \frac{\ln \bar{S}/K + [\bar{r} - (1/2)\sigma^2 + (2/3)c^2(T-t)^2 + \sigma c(T-t)](T-t) + B(t,T)}{[(1/3)c^2(T-t)^2 + \sigma c(T-t) + \sigma^2]^{1/2}\sqrt{T-t}}$$

Notice that in the original *Ho-Lee* model, b has a more concrete form

$$b(s) = F_s(0, s) + c^2 s$$

where $F(t, s)$ is the instantaneous forward rate.

CASE 4.3.4 *The Hull-White Model.* The short rate process r is given by

$$dr(s) = [\beta(s) - \alpha r(s)]ds + c d\hat{W}(s)$$

Assume that c, σ are both constant. Then

$$\varphi(s) = \exp[-\alpha(s - t)]$$

$$h(s) = \int_t^s e^{\alpha(\bar{s}-t)}\beta(\bar{s})d\bar{s}$$

$$g(s) = \frac{\sigma c}{\alpha}[e^{\alpha(s-t)} - 1]$$

$$f(s) = \frac{c^2}{2\alpha^2}[e^{\alpha(s-t)} - 2 + e^{-\alpha(s-t)}]$$

We have

$$P(t, T, \bar{r}) = \exp\{H(t,T, \bar{r})\}$$

with

$$H(t,T,\bar{r}) = \int_t^T \varphi(s)[f(s) - h(s) - \bar{r}]ds = \frac{c^2}{2\alpha^2}(T-t) + \left(\frac{c^2}{\alpha^3} + \frac{\bar{r}}{\alpha}\right)A(t,T) - \frac{c^2}{4\alpha^3}B(t,T) - C(t,T)$$

where $A(t, T)$, $B(t, T)$ are as in **CASE 4.3.2** and

$$C(t,T) = \int_t^T e^{-\alpha(s-t)} \left(\int_t^s e^{\alpha(\bar{s}-t)} \beta(\bar{s}) \mathrm{d}\bar{s} \right) \mathrm{d}s$$

Also,

$$d_1 = \frac{\ln(\bar{S}/K) + \left(\frac{6c^2}{\alpha^3} - \frac{\bar{r}}{\alpha} + \frac{4\sigma c}{\alpha^2}\right) A(t,T) - \frac{3c^2}{2\alpha^3} B(t,T) + C(t,T) + \left(\frac{1}{2}\sigma^2 + \frac{3c^2}{\alpha^2} + \frac{4\sigma c}{\alpha}\right)(T-t)}{\left\{\left(\frac{c^2}{\alpha^2} + \frac{2\sigma c}{\alpha} + \sigma^2\right)(T-t) + \left(\frac{2c^2}{\alpha^3} + \frac{2\sigma c}{\alpha^2}\right) A(t,T) - \frac{c^2}{2\alpha^2} B(t,T)\right\}^{1/2}}$$

$$d_2 = \frac{\ln(\bar{S}/K) + \left(\frac{4c^2}{\alpha^3} - \frac{\bar{r}}{\alpha} + \frac{2\sigma c}{\alpha^2}\right) A(t,T) - \frac{c^2}{\alpha^3} B(t,T) + C(t,T) + \left(-\frac{1}{2}\sigma^2 + \frac{2c^2}{\alpha^2} + \frac{2\sigma c}{\alpha}\right)(T-t)}{\left\{\left(\frac{c^2}{\alpha^2} + \frac{2\sigma c}{\alpha} + \sigma^2\right)(T-t) + \left(\frac{2c^2}{\alpha^3} + \frac{2\sigma c}{\alpha^2}\right) A(t,T) - \frac{c^2}{2\alpha^2} B(t,T)\right\}^{1/2}}$$

Appendix A

The key to recursively solving out the dynamic equilibrium model is the property that at each iteration step, the equilibrium price is shown to be a linear function of the current dividend earning.

Lemma A.1 Let f_i: $\mathbb{R} \to \mathbb{R}$ be a strictly monotone and continuous function, s.t. $f_i(x) = 0$ has a solution, $i = 1, 2, ..., m$. Let A be a nonempty subset of $\mathbb{R}_{++}$. Let $a_i > 0$, b_i be real constants, $i = 1, 2, ..., m$. Then the following equation

$$0 = \Sigma_i f_i\left(\frac{a_i x}{y} - b_i\right), x \in A$$

Uniquely determines y as a linear function of x on A: $y = a \cdot x$ where a is the unique nonzero constant s.t. $\Sigma_i f_i\left(\frac{a_i}{a} - b_i\right) = 0$

proof: The usual implicit function theorem is not applicable since A is not necessary an interval. But proof is remarkably simple. First it's easy to see that there is a unique number a such that

$$\Sigma_i f_i\left(\frac{a_i}{a} - b_i\right) = 0.$$

This is because when $\frac{1}{y}$ is sufficiently large $\Sigma_i f_i\left(\frac{a_i}{y} - b_i\right)$ is positive, and when $\frac{1}{y}$ is sufficiently negative, $\Sigma_i f_i\left(\frac{a_i}{y} - b_i\right)$ is negative due to the property that f_i is strictly monotone and $f_i(x) = 0$ has a solution x_i. The intermediate-value theorem guarantees the existence of a. Let $y = ax$. Then clearly y solves the equation. Since f_i is strictly monotone, $y = ax$ is the unique solution

QED

<u>Proof of **Theorem 2.4.4**</u>: Let $\{S_0, \ldots, S_{T-t}\}$ be any nonzero equilibrium price process. We show that $\{S_0, \ldots, S_{T-t}\}$ and investors policy together with their value functions are as given in the theorem. The result is easily seen to be true for period $T-1$ by deriving the **FOC** and using **Lemma A.1**. Now suppose it is true for $T-t$ so that

$$V^i{}_{T-t}(X, D) = \left(\sum_{k=0}^{t} \delta_i^{\,k}\right)(G^t D)^{\mathrm{T}} X + f^i_{T-t}(\varepsilon),\ \forall\, i$$

where $f^i_{T-t}(\varepsilon)$ contains no stochastic elements. Let S_{T-t-1} be the equilibrium price in period $T-t-1$. Then investor i's problem in period $T-t-1$ is

$$V^i{}_{T-t-1}(X, D) = \max_{Y^i_{T-t-1}} \{D^{\mathrm{T}}X - S_{T-t-1}\cdot(Y^i_{T-t-1} - X) -$$
$$-\sum_n \varepsilon^{(n)}_{T-t-1}\cdot q^i_n\,[S^{(n)}_{T-t-1}\cdot(Y^{(n),i}_{T-t-1} - X^{(n)})] +$$

$$+\delta_i\cdot\int_Z V^i{}_{T-t}(Y^i_{T-t-1}, \bar{D})\,P_{T-t-1,T-t}(D, \mathrm{d}\bar{D}) = \text{(by induction hypothesis)}$$

$$= \max_{Y^i_{T-t-1}} \{D^{\mathrm{T}}X - S_{T-t-1}\cdot(Y^i_{T-t-1} - X) - \sum_n \varepsilon^{(n)}_{T-t-1}\cdot q^i_n\,[S^{(n)}_{T-t-1}\cdot(Y^{(n),i}_{T-t-1} - X^{(n)})]$$
$$+ \left(\sum_{k=0}^{t} \delta_i^{\,k+1}\right)[G^{(t+1)}D]^{\mathrm{T}}\, Y^i_{T-t-1} + \delta_i f^i_{T-t}(\varepsilon)$$

FOC: differentiating w.r.t. $Y^{(n),i}_{T-t-1}$, for each n, yields

$$-S^{(n)}_{T-t-1} - \varepsilon_n \cdot S^{(n)}_{T-t-1} \cdot q^{i}_{n}{}' [S^{(n)}_{T-t-1} \cdot (Y^{(n),i}_{T-t-1} - X^{(n)})] + \left(\sum_{k=0}^{t} \delta_i^{k+1}\right) \sum_{m=1}^{N} g^{(t+1)}_{nm} D^{(m)} = 0 \quad \Rightarrow$$

$$S^{(n)}_{T-t-1} \cdot (Y^{(n),i}_{T-t-1} - X^{(n)}) = r^{i}_{n} \left(\frac{1}{\varepsilon^{(n)}_{T-t-1}} \cdot \left(\frac{(\sum_{k=0}^{t} \delta_i^{k+1}) \sum_{m=1}^{N} g^{(t+1)}_{nm} D^{(m)}}{S^{(n)}_{T-t-1}} - 1 \right) \right)$$

(A.1)

Adding across i gives

$$0 = \sum_i r^{i}_{n} \left(\frac{1}{\varepsilon^{(n)}_{T-t-1}} \cdot \left(\frac{(\sum_{k=0}^{t} \delta_i^{k+1}) \sum_{m=1}^{N} g^{(t+1)}_{nm} D^{(m)}}{S^{(n)}_{T-t-1}} - 1 \right) \right)$$

By **Lemma A.1**, $S^{(n)}_{T-t-1} = \dfrac{\sum_{m=1}^{N} g^{(t+1)}_{nm} D^{(m)}}{w^{(n)}_{T-t-1}}$, where $w^{(n)}_{T-t-1}$ is defined by

$$0 = \sum_i r^{i}_{n} \left(\frac{1}{\varepsilon^{(n)}_{T-t-1}} \cdot \left(w^{(n)}_{T-t-1} \cdot \sum_{k=0}^{t} \delta_i^{k+1} - 1 \right) \right)$$

Now substitute the expression $S^{(n)}_{T-t-1} = \dfrac{1}{w^{(n)}_{T-t-1}} \sum_{m=1}^{N} g^{(t+1)}_{nm} D^{(m)}$ back to (A.1) and we get

$$S^{(n)}_{T-t-1}\cdot(Y^{(n),i}_{T-t-1}-X^{(n)})=r^i_n\left(\frac{1}{\varepsilon^{(n)}_{T-t-1}}\cdot\left(w^{(n)}_{T-t-1}\sum_{k=0}^{t}\delta_i^{k+1}-1\right)\right) \quad \text{(A.2)}$$

and

$$Y^{(n),i}_{T-t-1}-X^{(n)}=w^{(n)}_{T-t-1}\cdot r^i_n\left(\frac{1}{\varepsilon^{(n)}_{T-t-1}}\cdot\left(w^{(n)}_{T-t-1}\cdot\sum_{k=0}^{t}\delta_i^{k+1}-1\right)\right) \quad \text{(A.3)}$$

Now Substituting (A.2) back into the expression for $V^i_{T-t-1}(X, D)$ gives the desired result. Hence induction is complete and necessity is proved. Finally since **FOC** is sufficient, the entire proof is complete.

QED

Proof of **Lemma 2.4.6**

We prove case of structural bias in buyers' favor only. We first show that $w^{(n)}_{T-t}(\theta)$ is locally strictly decreasing in θ, i.e. for every $\theta > 0$, there is some $\alpha > 0$ s.t. for every $\theta' \in (\theta, \theta + \alpha)$, $w^{(n)}_{T-t}(\theta') < w^{(n)}_{T-t}(\theta)$.

Let $\theta > 0$ be given with

$$\Sigma_i r^i_n\,[\theta\cdot(\alpha^i_{T-t}\cdot w^{(n)}_{T-t}(\theta) - 1)] = 0$$

Clearly, $\alpha^i_{T-t}\cdot w^{(n)}_{T-t}(\theta) - 1$ cannot be all zero (and therefore some must be positive and some must be negative) since α^i_{T-t}'s are distinct. By the definition of structural bias in buyers' favor, an infinitesimal increase of θ to the value θ' makes

$$\Sigma_i r_n^i \, [\theta' \cdot (\alpha_{T-t}^i \cdot w_{T-t}^{(n)}(\theta) - 1)] > 0 \qquad (**)$$

Since r_n^i is strictly increasing, the only way to achieve a new balance is for $w_{T-t}^{(n)}(\theta)$ to decrease to a lower level $w_{T-t}^{(n)}(\theta')$ so that the positive terms in (**) become less positive (i.e. decreased) and the negative terms become more negative in order to get the following

$$\Sigma_i r_n^i \, [\theta' \cdot (\alpha_{T-t}^i \cdot w_{T-t}^{(n)}(\theta') - 1)] = 0$$

Hence $w_{T-t}^{(n)}$ is locally strictly decreasing in θ. Since $w_{T-t}^{(n)}$ is a continuous function of θ, it follows now that $w_{T-t}^{(n)}$ is actually globally strictly decreasing in θ as well.

QED

Proof **Theorem 2.4.8**

To prove this in our framework, observe that in equilibrium, trade volume in stock n is the sum of all net buyers' demand $Y_{T-t}^{(n),i} - X_{T-t}^{(n),i} > 0$. It is also the absolute value of the sum of all net sellers' supply $Y_{T-t}^{(n),i} - X_{T-t}^{(n),i} < 0$. By (2.4.3) we have

$$Y_{T-t}^{(n),i} - X_{T-t}^{(n),i} = \frac{1}{S_{T-t}^{(n)}} \cdot r_n^i \left(\frac{1}{\varepsilon_{T-t}^{(n)}} \cdot \left(\frac{D_{T-t}^{(n)} \sum_{k=1}^{t} \delta_i^k}{S_{T-t}^{(n)}} - 1 \right) \right) \qquad (2.4.3)$$

First suppose the market is in equilibrium so $S_{T-t}^{(n)}$ is the equilibrium price. Let Let d be the equilibrium demand and s be the equilibrium supply and we have $d = s$, which represents the equilibrium trade volume. Now let$\varepsilon_{T-t}^{(n)}$ increase to a higher level $\varepsilon'^{(n)}_{T-t}$. The both aggregate demand and aggregate supply (in absolute value) must decrease by (2.4.3) (before price adjustment). Let d' be the new aggregate demand and let s' be the new aggregate supply (in absolute value). One of the three cases must occur.

Case 1. $d' = s'$ and thus the equilibrium price stays unchanged with equilibrium trade volume $d' < d$.

Case 2. $d' > s'$. The equilibrium price but adjust to a higher level so that d' is further decreased to a level d'' and s' is increased to a higher level s'' until $d'' = s''$ holds. In this case, the new equilibrium trade volume is $d'' < d' < d$.

Case 3. $d' < s'$. The equilibrium price must adjust down to a lower level further depressing s' to a lower value s'' but increasing d' to a higher level d'' until $s'' = d''$. But again we have $s'' < s' < s = d =$ the original equilibrium trade volume.

QED

Appendix B

proof of Lemma 3.1.3:

Let $\alpha_t = \int_0^t \frac{1}{B_u} \mathrm{d}c_u$. Then we have

$$\int_0^t B_u \mathrm{d}\alpha_u = c_t.$$

Now the integration by parts formula yields

$$\alpha_t \cdot B_t - \int_0^t \alpha_{u-} \mathrm{d}B_u = c_t.$$

Hence

$$\alpha_{t-} \cdot B_t - \int_0^t \alpha_u \mathrm{d}B_u = c_{t-},\ 0 \le t \le T$$

Now first suppose $(\theta, \varphi) \in \Lambda(w, c)$. Let $\varphi' = \varphi + \alpha_-$. Clearly $\varphi' \in \mathcal{L}(B)$.

Now we have

$$\theta_t \cdot S_t + \varphi_t' \cdot B_t = \theta_t \cdot S_t + \varphi_t \cdot B_t + \alpha_{t-} \cdot B_t = w + \int_0^t \theta_u \cdot \mathrm{d}G_u +$$

$$\int_0^t \varphi_u \mathrm{d}B_u - c_{t-} + c_{t-} + \int_0^t \alpha_u \mathrm{d}B_u = w + \int_0^t \theta_u \cdot \mathrm{d}G_u + \int_0^t \varphi_u' \mathrm{d}B_u,\ 0 \le t \le T$$

with

$$\theta_T \cdot S_T + \varphi_T' \cdot B_T = \Delta c_T + B_T \int_0^{T-} \frac{1}{B_u} \mathrm{d}c_u = B_T \int_0^T \frac{1}{B_u} \mathrm{d}c_u$$

Hence (3.1.4) is proved. Conversely, if (θ, φ') satisfy (1.4), then let

$$\varphi = \varphi' - \alpha\_$$

and we have (3.1.3).

QED

<u>proof of **Fact 3.1.5**</u>:

$\hat{M}^{(k)} = \Gamma(M^{(k)})$, $\forall\, k$. To see that $\hat{M}$ has the representation property (under Q), let $\hat{N}$ be a local F-martingale under Q. Then by **Fact 3.1.2**, $\hat{N}$ arises as the *Girsanov* transform of some local F- martingale N under P, i.e.

$$\hat{N} = N + \langle N, \int_0^t \eta_u \cdot \mathrm{d}M_u \rangle$$

Thus there is some $\theta \in \mathcal{L}_K^2(M)$ $(= \mathcal{L}_K^2(\hat{M}))$ s.t.

$$N_t = N_0 + \int_0^t \theta_u \cdot \mathrm{d}M_u = \hat{N}_0 + \int_0^t \theta_u \cdot \mathrm{d}M_u =$$

$$= \hat{N}_0 + \int_0^t \theta_u \cdot \mathrm{d}\hat{M}_u - \Sigma_k \int_0^t \theta_u^{(k)} \cdot \eta_u^{(k)} \cdot \mathrm{d}\langle M^{(k)} \rangle_u \Rightarrow$$

$$\hat{N}(t) = \hat{N}_0 + \int_0^t \theta_u \cdot \mathrm{d}\hat{M}_u - \Sigma_k \int_0^t \theta_u^{(k)} \eta_u^{(k)} \cdot \mathrm{d}\langle M^{(k)} \rangle_u + \langle N, \int_0^t \eta_u \cdot \mathrm{d}M_u \rangle$$

$$= \hat{N}_0 + \int_0^t \theta_u \cdot \mathrm{d}\hat{M}_u - \Sigma_k \int_0^t \theta_u^{(k)} \eta_u^{(k)} \cdot \mathrm{d}\langle M^{(k)} \rangle_u + \langle \int \theta_u \cdot \mathrm{d}M_u, \int \eta_u \cdot \mathrm{d}M_u \rangle$$

$$= \hat{N}_0 + \int_0^t \theta_u \cdot \mathrm{d}\hat{M}_u$$

Hence the representation property of $\hat{M}$ is established.

QED

proof of **Prop 3.1.6**:

Construction of Q and the representation property of $\hat{M}$ is given in **Fact 3.1.2**. Direct verification shows that

$$d\hat{G} = \hat{b} \cdot d\hat{M}.$$

To see that $\hat{G}$ is actually a martingale under Q, notice that by assumption, $\hat{b} \in \mathcal{H}^2_{N\times K}(\langle M\rangle;\ P)$. But since the density $Z_T = dQ/dP$ is square-integrable under P, it follows that $\hat{b} \in \mathcal{H}_{N\times K}(\langle M\rangle;\ Q) = \mathcal{H}_{N\times K}(\langle \hat{M}\rangle;\ Q)$ and thus $\hat{G}$ is a martingale under Q. Next we prove that the asset market is complete. Let $c \in \mathcal{L}^2(P)$. Let

$$\hat{w} = w/B_0 = E^Q\left(\int_0^T \frac{1}{B_u} dc_u\right) \text{ and } N = \{N_t\}_{t\in[0,\,T]} \text{ be given by}$$

$$N_t = E^Q\left[\int_0^T \frac{1}{B_u} dc_u \mid \mathcal{F}_t\right],\ 0 \le t \le T$$

Then N is a Q-martingale and therefore by the representation property of $\hat{M}$, there is some $v^{\mathsf{T}} \in \mathcal{H}_{1\times K}(\langle \hat{M}\rangle) =$ s.t.

$$N(t) = w/B_0 + \int_0^t v_u \cdot d\hat{M}_u,\ 0 \le t \le T.$$

Let θ satisfy $\theta b = v$. Then we have $\theta \in \mathcal{H}(\langle \hat{G} \rangle; Q)$. Let $\varphi = N - \theta \cdot \hat{S}$. Then clearly $\varphi \in \mathcal{L}(B)$ and

$$\theta_t \cdot (\hat{S}_t + \Delta \hat{D}_t) + \varphi_t = N_t = \hat{w} + \int_0^t v_u \cdot d\hat{M}_u =$$

$$= \hat{w} + \int_0^t \theta \cdot d\hat{G}, \ 0 \le t \le T, \quad \text{and}$$

$$\theta_T \cdot (\hat{S}_T + \Delta \hat{D}_T) + \varphi_T = N_T = \int_0^T \frac{1}{B_u} dc_u .$$

It follows from **Lemma 3.1.4** that the market (S, B, D) is complete w.r.t $(\Theta, \mathcal{L}^2(P))$.

QED

proof of **Cor 3.1.7**:

Suppose there is some *CCF* process c in $\mathcal{L}^2(P)$ that is financed by some $(\theta, \varphi) \in \Theta$ with some initial investment w s.t. $c + w \succ 0$ under the original markets (S, B, D). Then by the numeraire invariance theorem, $\hat{c}$ is financed by (θ, φ) under $(S/B, 1, \hat{D})$ with initial investment $\hat{w} = w/B_0$, where

$$\hat{c} = \int \frac{1}{B} dc .$$

Thus we have

$$\hat{w} + \hat{c}_T = \int_0^T \theta_u d\hat{G}_u$$

Since $\int \theta \, \mathrm{d}\hat{G}$ is a martingale under Q, it follows that

$$0 = E^Q[\hat{w} + \hat{c}_T]$$

in direct contradiction to $\hat{w} + \hat{c} \succ 0$. Hence, Θ contains no arbitrage. Next let $c \in \mathcal{L}^2(P)$ be any *CCF* process so that $\hat{c} := \int \frac{1}{B} \mathrm{d}c$ is financed by some (θ, φ) in Θ. We must then have

$$\theta_t \cdot S_t / B_t + \varphi_t = \hat{c}_T - \hat{c}_{t-} - \int_t^T \theta_u \, \mathrm{d}\hat{G}_u$$

Taking expectation under Q conditional on $\mathcal{F}_t$ gives

$$\theta_t \cdot S_t / B_t + \varphi_t = E^Q[\hat{c}_T - \hat{c}_{t-} \mid \mathcal{F}_t]$$

QED

Appendix C

<u>**proof of Lemma 4.2.2**</u>:

First, let $F(\cdot\ ; m, b^2)$ denotes the c.d.f of the normal distribution $N(m, b^2)$. Then we have

$$1 - F(x;\ m, b^2) = \Phi(-\frac{x-m}{b})$$

Next, the joint density of $z(T)$ and $y(t) - y(T)$ is

$f(z,y)$ =

$$= \frac{1}{2\pi b_1 b_2 \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(z-a_1)^2}{b_1^2} - \frac{2\rho(z-a_1)(y-a_2)}{b_1 b_2} + \frac{(y-a_2)^2}{b_2^2} \right] \right\}$$

Now let

$V_1 = E^Q[1_{\{z(T) \geq \ln K\}} \cdot e^{z(T) + [y(t) - y(T)]} \mid \mathcal{F}_t]$, and $V_2 = E^Q[1_{\{z(T) \geq \ln K\}} \cdot e^{y(t) - y(T)} . K \mid \mathcal{F}_t]$

Then

$$V_1 = \int_{\ln K}^{\infty} \int_{-\infty}^{\infty} e^{z+y} f(z,y) \mathrm{d}y \mathrm{d}z = \int_{\ln K}^{\infty} e^{z} \left[\int_{-\infty}^{\infty} e^{y} f(z,y) \mathrm{d}y \right] \mathrm{d}z$$

To computer the inner integral in the square bracket, let

$$u = \frac{y - a_2}{b_2}, \quad v = \frac{z - a_1}{b_1}$$

Then

$$\int_{-\infty}^{\infty} e^y f(z,y)\mathrm{d}y =$$

$$= \frac{1}{\sqrt{2\pi}\ b_1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left\{b_2 u + a_2 - \frac{1}{2(1-\rho^2)}[u^2 - 2\rho uv + v^2]\right\}\mathrm{d}u$$

$$=$$

$$\frac{1}{\sqrt{2\pi}\ b_1}$$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left\{-\frac{u^2 - 2[\rho v + (1-\rho^2)b_2]u + v^2 - 2(1-\rho^2)a_2}{2(1-\rho^2)}\right\}\mathrm{d}u$$

$$= \frac{1}{\sqrt{2\pi}\ b_1}\int_{\ln K}^{\infty}\exp\left\{-\frac{[v-(\rho b_2 + b_1)]^2 - (b_1^2 + 2\rho b_1 b_2 + b_2^2) - 2(a_1 + a_2)}{2}\right\}\mathrm{d}z =$$

$$= \frac{1}{\sqrt{2\pi}\ b_1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\{h(u)\}\mathrm{d}u$$

where

$$h(u) = -\frac{[u - \rho v - (1-\rho^2)b_2]^2 - [\rho v + (1-\rho^2)b_2]^2 + v^2 - 2(1-\rho^2)a_2}{2(1-\rho^2)}$$

Thus the inner integral gives

$$\frac{1}{\sqrt{2\pi}\ b_1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\{h(u)\}du=$$
$$\frac{1}{\sqrt{2\pi}\ b_1}\exp\left(-\frac{v^2-2\rho\, b_2 v-(1-\rho^2)b_2^2-2a_2}{2}\right)$$

$\Rightarrow$

$$V_1=\frac{1}{\sqrt{2\pi}\ b_1}\int_{\ln K}^{\infty}\exp\left\{-\frac{v^2-2\rho\, b_2 v-2b_1 v+\rho^2 b_2^2-2a_1-b_2^2-2a_2}{2}\right\}=$$

$$=\exp\left\{\frac{(b_1^2+2\rho\, b_1 b_2+b_2^2)+2(a_1+a_2)}{2}\right\}[1-F(\ln K;\ a_1+\rho b_1 b_2+b_1^2,\ \ b_1^2)]=$$

$$=\exp\left\{\frac{(b_1^2+2\rho\, b_1 b_2+b_2^2)+2(a_1+a_2)}{2}\right\}\Phi\left(-\frac{\ln K-a_1-\rho\, b_1 b_2-b_1^2}{b_1}\right)=$$

$$=\exp\left\{\frac{(b_1^2+2b_{12}+b_2^2)+2(a_1+a_2)}{2}\right\}\Phi\left(-\frac{\ln K-a_1-b_{12}-b_1^2}{b_1}\right)$$

Similarly, $\frac{1}{\sqrt{2\pi}\,b_1}\exp\left(-\frac{v^2-2\rho b_2 v-(1-\rho^2)b_2^2-2a_2}{2}\right)$

$$V_2=\frac{1}{\sqrt{2\pi}\,b_1}K\int_{\ln K}^{\infty}\exp\left\{-\frac{(v-\rho b_2)^2-b_2^2-2a_2}{2}\right\}dz=$$

$$=\exp\left\{\frac{b_2^2+2a_2}{2}\right\}\frac{1}{\sqrt{2\pi}\,b_1}\int_{\ln K}^{\infty}\exp\left\{-\frac{[z-(a_1+\rho b_1\cdot b_2)]^2}{2b_1^2}\right\}dz=$$

$$=\exp\left\{\frac{1}{2}b_2^2+a_2\right\}K\cdot[1-F(\ln K;\ a_1+\rho b_1\cdot b_2,\ b_1^2)]=$$

$$=\exp\left\{\frac{1}{2}b_2^2+a_2\right\}K\cdot\Phi\left(-\frac{\ln K-a_1-b_{12}}{b_1}\right)$$

Hence

$$V(t)=\exp\left\{\frac{1}{2}(b_1^2+2b_{12}+b_2^2)+(a_1+a_2)\right\}\Phi(d_1)-\exp\left\{\frac{1}{2}b_2^2+a_2\right\}K\cdot\Phi(d_2)$$

where

$$d_1=-\frac{\ln K-b_{12}-b_1^2}{b_1},\quad d_2=-\frac{\ln K-a_1-b_{12}}{b_1}$$

Finally, notice that

$$\exp\left\{\frac{1}{2}(b_1^2 + 2b_{12} + b_2^2) + (a_1 + a_2)\right\}$$

is exactly the moment generating function $g(\lambda)$ of the random variable $y(t) - y(T) + z(T)$ computed at $\lambda = 1$. In other words

$$\exp\left\{\frac{1}{2}(b_1^2 + 2b_{12} + b_2^2) + (a_1 + a_2)\right\} = E^Q[e^{z(T) + y(t) - y(T)} \mid X(t)] = (\textit{Markov}$$

property)

$$= E^Q[e^{z(T) + y(t) - y(T)} \mid \mathcal{F}_t] = E[S(T) \cdot e^{-\int_t^T r(\tau)\mathrm{d}\tau} \mid \mathcal{F}_t] = S(t)$$

by (4.1.2). Also,

$$\exp\left\{\frac{1}{2}b_2^2 + a_2\right\}$$

is the moment generating function $h(\tau)$ of $y(t) - y(T)$ evaluated at $\tau = 1$ and so by (4.1.3)

$$\exp\left\{\frac{1}{2}b_2^2 + a_2\right\} = E^Q[e^{-\int_t^T r(\tau)\mathrm{d}\tau} \mid \mathcal{F}_t] = P(t, T)$$

QED

<u>proof of **Lemma 4.2.3**</u>:

Here we demonstrate only how to derive formulas for a_2 and b_2. Derivations for other parameters are the same. Consider the system of linear differential equations (4.2.7). Solving the linear equation for m yields

$$m_1(s) = e^{\int_t^s a(\tau)\mathrm{d}\tau}\,[r(t) + \int_t^s e^{-\int_t^{\bar{t}} a(\tau)\mathrm{d}\tau}\cdot b(\bar{t})\mathrm{d}\bar{t}]$$

Next, $m_2(s)$ satisfies

$$\mathrm{d}m_2(s) = m_1(s)\mathrm{d}s\,,\ \text{s.t. } m_2(t) = y(t) \Rightarrow$$

$$m_2(s) = y(t) + \int_t^s m_1(\tau)\mathrm{d}\tau =$$

$$= y(t) + r(t)\int_t^s e^{\int_t^{\bar{s}} a(\tau)\mathrm{d}\tau}\mathrm{d}\bar{s} - \int_t^s e^{\int_t^{\bar{s}} a(\tau)\mathrm{d}\tau}\left[\int_t^{\bar{s}} e^{-\int_t^{\bar{t}} a(\tau)\mathrm{d}\tau}\cdot b(\bar{t})\mathrm{d}\bar{t}\right]\mathrm{d}\bar{s}$$

Now

$$a_2 = y(t) - m_2(T) =$$

$$= -r(t)\int_t^T e^{\int_t^s a(\tau)\mathrm{d}\tau}\mathrm{d}s - \int_t^T e^{\int_t^s a(\tau)\mathrm{d}\tau}\left[\int_t^s e^{-\int_t^{\bar{t}} a(\tau)\mathrm{d}\tau}\cdot b(\bar{t})\mathrm{d}\bar{t}\right]\mathrm{d}s$$

To compute variance and covariance, let

$$\Sigma_2 = \begin{pmatrix} \sigma_r^2 & \sigma_{r,y} \\ \sigma_{r,y} & \sigma_y^2 \end{pmatrix}$$ Then

$$d\Sigma_2/ds = A_2\Sigma_2 + \Sigma_2 A_2^{\mathrm{T}} + C_2 C_2^{\mathrm{T}}$$

The above reduces to a system of linear differential equations in three unknowns.

$$d\sigma_r^2/ds = 2a\sigma_r^2 + c^2$$
$$d\sigma_{r,y}/ds = a\sigma_{r,y} + \sigma_r^2$$
$$d\sigma_y^2/ds = 2\sigma_{r,y}$$

We get recursively,

$$\sigma_r^2(s) = e^{2\int_t^s a(\tau)d\tau} \cdot \int_t^s e^{-2\int_t^{\bar{s}} a(\tau)d\tau} \cdot c^2(\bar{s})d\bar{s} = \varphi^2(s)\int_t^s \left[\frac{c(\bar{s})}{\varphi(\bar{s})}\right]^2 d\bar{s}$$

$$\sigma_{r,y}(s) = \varphi(s)\cdot \int_t^s \left[\varphi(\bar{s})\cdot \int_t^{\bar{s}} \left[\frac{c(\bar{t})}{\varphi(\bar{t})}\right]^2 d\bar{t}\right] d\bar{s} = \varphi(s)f(s)$$ where

$$f(s) = \int_t^s \left[\varphi(\bar{s})\cdot \int_t^{\bar{s}} \left[\frac{c(\bar{t})}{\varphi(\bar{t})}\right]^2 d\bar{t}\right] d\bar{s}$$

$$\sigma_y^2(s) = 2\int_t^s \varphi(\bar{s}) f(\bar{s})d\bar{s}$$ Finally, notice that

$$a_2 = b_2^2 = \sigma_y^2(T)$$

QED

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